Numerical evaluation of dilogarithm $\mathrm{Li}_2(z)$

— Osácar, Palacián, Palacios,
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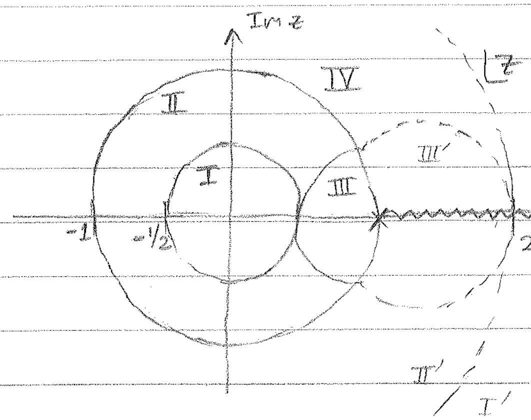
$\boxed{\text{Put } z = re^{i\theta}}$

Divide complex plane into 4 regions:



In region I: $0 \le r \le 1/2$

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$$

In region II: $1/2 < r \le 1$ (except radius 1/2 disk around $z=0$)

$$\mathrm{Li}_2(z) = -\int_0^1 dt\, \frac{\ln(1-zt)}{t}$$

approximated by Gaussian quadrature

In region III:

$$\mathrm{Li}_2(z) = \underbrace{-\mathrm{Li}_2(1-z)}_{\text{evaluated in region I}} - \ln(z)\ln(1-z) + \frac{\pi^2}{6}$$

$1-z = 1-re^{i\theta}$

abs.: $\sqrt{1-2r\cos\theta+r^2}$

arg: $\tan^{-1}\left(\frac{-r\sin\theta}{1-r\cos\theta}\right)$

In region IV

$$\mathrm{Li}_2(z) = -\mathrm{Li}_2\left(\frac{1}{z}\right) - \frac{1}{2}\ln^2(-z) - \frac{\pi^2}{6}$$

maps to region I, II, or III

(images shown as I', II', III')