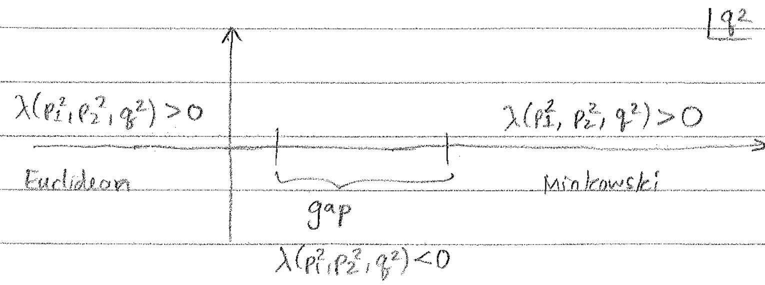
Notes on numerical evaluation of C_0

For real masses and momenta, can split evaluation into real and imaginary parts.



$\mathrm{Li}_2(z)$ is a real analytic function on $z \leq 1$.

$$\therefore \quad [\mathrm{Li}_2(z)]^* = \mathrm{Li}_2(z^*)$$

or $\mathrm{Li}_2(x+iy) + \mathrm{Li}_2(x-iy)$ is purely real

In Eucl. & Mink region, $\boxed{\lambda(p_1^2, p_2^2, q^2) > 0}$,

z_i are purely real

$z_{i\pm}$ are real if $\lambda(p_i^2, m_A^2, m_B^2) > 0$

are complex conjugates if $\lambda(p_i^2, m_A^2, m_B^2) < 0$

So,

$$\mathrm{Re}\, C_0 = \frac{1}{\sqrt{\lambda(p_1^2, p_2^2, q^2)}} \sum_{i=\{1,2,12\}}' \left[\underbrace{\mathrm{Li}_2\left(\frac{1-z_i}{z_{i+}-z_i}\right) + \mathrm{Li}_2\left(\frac{1-z_i}{z_{i-}-z_i}\right)}_{} - \mathrm{Li}_2\left(\frac{-z_i}{z_{i+}-z_i}\right) - \mathrm{Li}_2\left(\frac{-z_i}{z_{i-}-z_i}\right) \right]$$

If $\lambda(p_i^2, m_A^2, m_B^2) < 0$
imaginary parts of Li_2 cancel
$\Rightarrow$ combine.

If $\lambda(p_i^2, m_A^2, m_B^2) > 0$,
arguments of $\mathrm{Li}_2(\ )$ are real.

If $\boxed{\lambda(q^2, p_1^2, p_2^2) < 0}$ (Gap region),

leading $\frac{1}{\sqrt{\lambda(p_1^2 p_2^2 q^2)}}$ becomes imaginary. $\therefore$ Need imaginary part of Li_2 functions.

Revisit calculation to account for additional logarithms:

$$I_i = \int_0^1 dz \, \frac{1}{z - z_i} \Big[\ln \tilde{\alpha} \underbrace{(z - z_{i+})(z - z_{i-})}_{\text{this has neg. im part}} - \ln \tilde{\alpha} \underbrace{(z_i - z_{i+})(z_i - z_{i-})}_{\text{Let neg. im part of this be } \delta.} \Big]$$

(is ϵ for $p_1^2 p_2^2, q^2, m_0, m_1, m_2 \in \mathbb{R}$)

$$= \int_0^1 dz \, \frac{1}{z - z_i} \Big[\ln(\tilde{\alpha} - i\epsilon) + \underbrace{\ln(z - z_{i+})(z - z_{i-})}_{\text{split}} - \ln(\tilde{\alpha} - i\delta) - \underbrace{\ln(z_i - z_{i+})(z_i - z_{i-})}_{\text{split}} \Big]$$

combine

$$= \int_0^1 dz \, \frac{1}{z - z_i} \Big[\ln(z - z_{i+}) + \ln(z - z_{i-}) - \eta(-z_{i+}, -z_{i-})$$
$$- \ln(z_i - z_{i+}) - \ln(z_i - z_{i-}) + \eta(z_i - z_{i+}, z_i - z_{i-})$$
$$- \eta\left(\tilde{\alpha} - i\epsilon, \frac{1}{\tilde{\alpha} - i\delta}\right) \Big]$$

leads to extra logs.

Use $\int_0^1 \frac{dz}{z - z_i} = \ln\left(\frac{1 - z_i}{-z_i}\right)$ to do integrals prop. to η.

$$I_i = \sum_{\pm} \int_0^1 dz \, \frac{\ln(z - z_{i\pm}) - \ln(z_i - z_{i\pm})}{z - z_{i\pm}}$$
$$- \ln\left(\frac{1 - z_i}{-z_i}\right) \left[\eta(-z_{+i}, -z_{i-}) - \eta(z_i - z_{i+}, z_i - z_{i-}) + \eta\left(\tilde{\alpha} - i\epsilon, \frac{1}{\tilde{\alpha} - i\delta}\right) \right]$$

First line is $R_\pm$.

$$I_i = \sum_{\pm} \Big[-Li_2\left(\frac{1 - z_i}{z_{i\pm} - z_i}\right) + Li_2\left(\frac{-z_i}{z_{i\pm} - z_i}\right)$$
$$+ \ln\left(\frac{1 - z_i}{z_{i\pm} - z_i}\right) \eta\left(1 - z_{i\pm}, \frac{1}{z_i - z_{i\pm}}\right) - \ln\left(\frac{-z_i}{z_{i\pm} - z_i}\right) \eta\left(-z_{i\pm}, \frac{1}{z_i - z_{i\pm}}\right) \Big]$$

$$- \ln\left(\frac{1 - z_i}{-z_i}\right) \Big[\underbrace{\eta(-z_{+i}, -z_{i-}) - \eta(z_i - z_{i+}, z_i - z_{i-})}_{\text{non-contributing for real params.}} + \underbrace{\eta\left(\tilde{\alpha} - i\epsilon, \frac{1}{\tilde{\alpha} - i\delta}\right)}_{\text{non contributing if } \tilde{\alpha} \in \mathbb{R}.} \Big]$$

Need imaginary parts of these functions to obtain real part of C_0