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Feynman parametric representation of one-loop scalar integrals in $d = D - 4\epsilon$ dimensions.

Definition:

$$T_N^0 = \left(\frac{i e^{-\gamma_E \epsilon}}{(4\pi)^{d/2}}\right)^{-1} \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{1}{D_1(k)\, D_2(k) \cdots D_N(k)}$$

Combine denominators following Feynman's trick:

$$= \left(\frac{i e^{\gamma_E \epsilon}}{(4\pi)^{d/2}}\right)^{-1} \int_0^1 dx_1 \cdots dx_N\, \delta\Big(1 - \sum_{i=1}^N x_i\Big)\, \Gamma(N) \underbrace{\mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{1}{[x_1 D_1 + \ldots + x_N D_N]^N}}_{\text{Define: } I_N^0(x_1, \ldots, x_N)}$$

$$\begin{aligned} \text{Denom} &= x_1 D_1 + \ldots x_N D_N \\ &= k^2 + 2k\cdot Q + A^2 \end{aligned}$$

put $\Delta = Q^2 - A^2$ — see next few pages for explicit expressions.

Integrate over k

$$= \left(\frac{i e^{\gamma_E \epsilon}}{(4\pi)^{d/2}}\right)^{-1} \int_0^1 dx_1 \cdots dx_N\, \delta\Big(1 - \sum_i x_i\Big)\, \cancel{\Gamma(N)}\, \mu^{2-\frac{d}{2}} \underbrace{\frac{(-1)^N i}{(4\pi)^{d/2}}}_{\text{factor out} \to \text{cancel.}} \frac{\Gamma(N - \frac{d}{2})}{\cancel{\Gamma(N)}} \left(\frac{1}{\Delta}\right)^{N - \frac{d}{2}}$$

$$= \mu^{2\epsilon} (-1)^N e^{\gamma_E \epsilon}\, \Gamma\Big(N - \frac{d}{2}\Big) \int_0^1 dx_1 \cdots dx_N\, \delta\Big(1 - \sum_i x_i\Big)\, \Delta^{d/2 - N}$$

Denominator function given generally: (proof on next pg)

$$\begin{aligned} \Delta &= - \sum_{\{i<j\}=1}^{N} x_i x_j (p_i - p_j)^2 + \sum_{i=1}^{N} x_i m_i^2 \\ &= \frac{1}{2} \sum_{i,j=1}^{N} x_i x_j \underbrace{\left[-(p_i - p_j)^2 + m_i^2 + m_j^2 - i\varepsilon\right]}_{= Y_{ij}^{(N)} \text{ "Modified Cayley matrix"}} \end{aligned}$$

$\therefore$

$$T_N^0 = \mu^{2\epsilon} (-1)^N e^{\gamma_E \epsilon}\, \Gamma\Big(N - \frac{d}{2}\Big) \times \int_0^1 dx_1 \cdots dx_N\, \delta\Big(1 - \sum_i x_i\Big) \Bigg(\underbrace{\frac{1}{2} \sum_{i,j=1}^{N} x_i x_j \left[-(p_i - p_j)^2 + m_i^2 + m_j^2 - i\varepsilon\right]}_{Q^2 - A^2}\Bigg)^{\frac{d}{2} - N}$$

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Explicit form for Δ (denominator polynomial)

$$Q^\mu = (x_1 p_1 + \ldots + x_N p_N)^\mu$$

$$A^2 = x_1(p_1^2 - m_1^2 + i\varepsilon) + \ldots + x_N(p_N^2 - m_N^2 + i\varepsilon)$$

Then:

$$\Delta = Q^2 - A^2$$

$$= \underbrace{[x_1 p_1 + \ldots + x_N p_N]^2}_{\text{organize by square \& cross terms}} - x_1(p_1^2 - m_1^2 + i\varepsilon) - \ldots - x_N(p_N^2 - m_N^2 + i\varepsilon)$$

$$= \Big[\overbrace{x_1^2 p_1^2 + \ldots + x_N^2 p_N^2}^{N \text{ square terms}} + \overbrace{2x_1x_2\, p_1\cdot p_2 + \ldots + 2x_{N-1}x_N\, p_{N-1}\cdot p_N}^{\frac{N^2-N}{N} \text{ cross terms}}\Big]$$
$$- x_1(p_1^2 - m_1^2 + i\varepsilon) - \ldots - x_N(p_N^2 - m_N^2 + i\varepsilon)$$

In square terms, replace one factor of x_i with $1 - x_1 - \ldots - \hat{x}_j \ldots - x_N$

$$= x_1(\cancel{1} - x_2 - \ldots - x_N)p_1^2 + x_2(\cancel{1} - x_1 - x_3 - \ldots - x_N)p_2^2 + \ldots + x_N(\cancel{1} - x_1 - \ldots - x_{N-1})p_N^2$$
$$+ 2x_1x_2\, p_1\cdot p_2 + \ldots + 2x_{N-1}x_N\, p_{N-1}\cdot p_N$$
$$- \cancel{x_1 p_1^2} - \cancel{x_2 p_2^2} - \ldots - \cancel{x_N p_N^2} + x_1 m_1^2 + \ldots + x_N m_N^2 - \underbrace{(x_1 + \ldots + x_N)}_{1}\, i\varepsilon$$

[very important for $i\varepsilon$ regularization]

All "1" terms are cancelled — the remaining terms in square terms are of the form $x_1x_2\,p_1^2, \ldots, x_{N-1}x_N\,p_N^2$. Collect like pairs of $x_i x_j$.

ORIGIN:

$$= (\overbrace{-x_1x_2\,p_1^2 - x_1x_2\,p_2^2}^{\text{square terms}} + \overbrace{2x_1x_2\,p_1\cdot p_2}^{\text{cross terms}}) + (\overbrace{-x_1x_3\,p_1^2 - x_1x_3\,p_3^2}^{\text{square terms}} + \overbrace{2x_1x_3\,p_1\cdot p_3}^{\text{cross terms}})$$
$$+ \ldots + (-x_{N-1}x_N\,p_{N-1}^2 - x_{N-1}x_N\,p_N^2 + 2x_{N-1}x_N\,p_{N-1}\cdot p_N) + x_1 m_1^2 + \ldots + x_N m_N^2 - i\varepsilon$$

$$= -x_1x_2(p_1 - p_2)^2 - x_1x_3(p_1 - p_3)^2 - \ldots - x_{N-1}x_N(p_{N-1} - p_N)^2$$
$$+ x_1 m_1^2 + \ldots + x_N m_N^2 - i\varepsilon$$

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$$\Delta = \sum_{i=1}^{N} \sum_{i<j}^{N} \left(-x_i x_j (p_i - p_j)^2\right) + \sum_{i=1}^{N} x_i m_i^2 + i\varepsilon$$

increase range to $0 \to N$
- diagonal terms $i=j$ vanish,
- off-diagonal terms double-counted
$\Rightarrow$ multiply by $1/2$.

Insert $1 = \sum_j x_j$

$$\Delta = \frac{1}{2} \sum_{i,j}^{N} \left(-x_i x_j (p_i - p_j)^2\right) + \sum_{i,j}^{N} \underbrace{x_i x_j}_{\text{symmetric}} \underbrace{m_j^2} + i\varepsilon$$

symmetrize: $m_j^2 \to \frac{1}{2}(m_i^2 + m_j^2)$

$$= \frac{1}{2} \sum_{i,j=1}^{N} x_i x_j \underbrace{\left[-(p_i - p_j)^2 + m_i^2 + m_j^2\right]}_{:= Y_{ij}} + i\varepsilon$$

(Modified Cayley matrix)