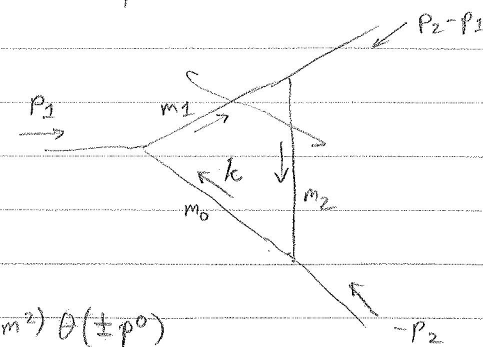
Imaginary part of $C_0(p_1^2, q^2, p_2^2; m_0, m_1, m_2)$

In physical region, $\lambda(p_1^2, p_2^2, q^2) > 0$ imaginary part is given by discontinuity across cut in all three channels p_1^2, p_2^2, q^2.



First: Discontinuity across cut in q^2 plane

Use Cutkosky's rule:

$$\xrightarrow{p} = \frac{1}{p^2 - m^2 - i\varepsilon} \Longrightarrow -2\pi i\, \delta(p^2 - m^2)\, \theta(\pm p^0)$$

sign fixed by flow of momentum though cut propagator.

$$\text{Disc}\begin{pmatrix}\text{loop}\\ \text{integral}\end{pmatrix} =$$

$$= \mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 - m_0^2 + i\varepsilon} (-2\pi i)\, \delta\big((k+p_1)^2 - m_1^2\big)\, \theta\big(-(k^0 - p_1^0)\big)\, (-2\pi i)\, \delta\big((k+p_2)^2 - m_2^2\big)\, \theta\big(k_2^0 + p_2^0\big)$$

$$-i \times -i = -1$$

Insert $\displaystyle\int \frac{d^dk_2}{(2\pi)^d} (2\pi)^d\, \delta^{(d)}\big(k_2 - (k+p_2)\big) = 1$

$\displaystyle\int \frac{d^dk_1}{(2\pi)^d} (2\pi)^d\, \delta^{(d)}\big(k_1 + (k+p_1)\big) = 1$

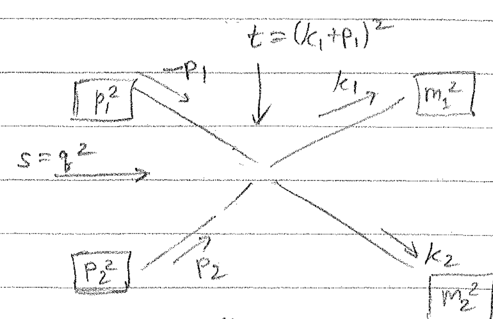
Integrate over k fixing $k = -k_1 - p_1$

$$= -\mu^{2\epsilon}\int \frac{d^dk_1}{(2\pi)^d} \frac{d^dk_2}{(2\pi)^d} \frac{1}{(k_1+p_1)^2 - m_0^2 + i\varepsilon}\, 2\pi\delta(k_1^2 - m_1^2)\, \theta(+k_1^0)$$

$$\times 2\pi\delta\big(\underbrace{(-k_1 - p_1 + p_2)}_{k_2}{}^2 - m_2^2\big)\, \theta\big(\underbrace{-k_1^0 - p_1^0 + p_2^0}_{+k_2^0}\big)\, (2\pi)^d \delta^{(d)}(k_2 + k_1 + p_1 - p_2)$$

$$= -\mu^{2\epsilon}\int \frac{d^dk_1}{(2\pi)^d}\, 2\pi\,\delta(k_1^2 - m_1^2)\, \theta(k_1^0) \int \frac{d^dk_2}{(2\pi)^d}\, 2\pi\,\delta(k_2^2 - m_2^2)\, \theta(k_2^0)\, (2\pi)^d \delta^{(d)}(k_2 + k_1 + p_1 - p_2) \frac{1}{(k_1+p_1)^2 - m_0^2 + i\varepsilon}$$

$$= -\mu^{2\epsilon}\int d(\text{LIPS}_2; q, k_1, k_2)_d\, \frac{1}{t - m_0^2 + i\varepsilon}$$



Effective process

For IR finite case, take $d \to 4$.

$$\mathrm{Disc}\begin{pmatrix}\text{loop}\\ \text{integral}\end{pmatrix} = -\int d(\mathrm{LIPS}_2\,;\,q\,;\,k_1,k_2)\,\frac{1}{t-m_0^2-i\varepsilon}$$

$$= -\left[\frac{1}{8\pi}\,\frac{1}{\lambda^{1/2}(p_1^2,p_2^2,q^2)}\int_{t_{back}}^{t_{forw}}\frac{dt}{t-m_0^2-i\varepsilon}\right]\theta\left(q^2-(m_1+m_2)^2\right)$$

$$= -\frac{1}{8\pi}\,\frac{1}{\lambda^{1/2}(p_1^2,p_2^2,q^2)}\,\ln\left(\frac{t_{forw}-m_0^2+i\varepsilon}{t_{back}-m_0^2+i\varepsilon}\right)\theta\left(q^2-(m_1+m_2)^2\right)$$

$$t_{\substack{forw\\ back}}\left(q^2;p_1^2,p_2^2;m_b^2,m_c^2\right) =$$

$$= \frac{1}{2q^2}\left[-(q^2)^2+q^2(p_1^2+p_2^2+m_1^2+m_2^2)-(p_1^2-p_2^2)(m_1^2-m_2^2)\right]$$

$$\pm \cancel{2}\,\frac{1}{\cancel{4}_{\,2}\,q^2}\,\lambda^{1/2}(p_1^2,p_2^2,q^2)\,\lambda^{1/2}(m_1^2,m_2^2,q^2)$$

$$= \frac{1}{2q^2}\left[-(q^2)+q^2(p_1^2+p_2^2+m_1^2+m_2^2)-(p_1^2-p_2^2)(m_1^2-m_2^2)\pm\lambda^{1/2}(p_1^2,p_2^2,q^2)\,\lambda^{1/2}(m_1^2,m_2^2,q^2)\right]$$

Sum over all channels:

$$\mathrm{Disc}\begin{pmatrix}\text{loop}\\ \text{integral}\end{pmatrix} = \frac{i}{16\pi^2}\left[\mathrm{Disc}_{q^2}(C_0)+\mathrm{Disc}_{p_1^2}(C_0)+\mathrm{Disc}_{p_2^2}(C_0)\right]$$

$$= \frac{i}{16\pi^2}\left[2i\,\mathrm{Im}\,C_0\right]$$

$$\Rightarrow \mathrm{Im}\,C_0 = -8\pi^2\left[\mathrm{Disc}_{q^2}\begin{pmatrix}\text{loop}\\ \text{integral}\end{pmatrix}+\mathrm{Disc}_{p_1^2}\left(\quad\right)+\mathrm{Disc}_{p_2^2}\left(\quad\right)\right]$$

$$= \frac{\pi}{\lambda^{1/2}(p_1^2,p_2^2,q^2)}\left\{\ln\left(\frac{t_{forw}-m_0^2+i\varepsilon}{t_{back}-m_0^2+i\varepsilon}\right)\theta\left(q^2-(m_1+m_2)^2\right)+(q\to p_1)+(q\to p_2)\right\}$$

$$\boxed{= \frac{\pi}{\lambda^{1/2}(p_1^2,p_2^2,q^2)}\sum_{i=\{1,2,12\}}\left[\ln\left(\frac{t_{forw}(p_a^2,p_b^2,p_c^2;m_b^2,m_c^2)-m_a^2+i\varepsilon}{t_{back}(p_a^2,p_b^2,p_c^2;m_b^2,m_c^2)-m_a^2+i\varepsilon}\right)\theta\left(p_a^2-(m_b+m_c)^2\right)\right] \quad \text{if } \lambda(p_1^2,p_2^2,q^2)>0}$$

In gap region, $\boxed{\lambda(p_1^2, p_2^2, q^2) < 0}$

need continuation to unphysical region.

— Norton & Fronsdal (1963) or Amati & Fubini (19)

Cutkosky's method: [has sign ambiguity resolved by experimental maths]

put $\dfrac{1}{k^2 - m_0^2 + i\varepsilon} \longrightarrow -2\pi i\, \delta(k^2 - m_0^2)\, \theta(k_0^0)$

$$\mathrm{Im}\, C_0 = -8\pi^2 \left[-\frac{1}{8\pi} \frac{1}{\lambda^{1/2}(p_1^2, p_2^2, q^2)} \int_{t_{back}}^{t_{forw}} -2\pi i\, \delta(t - m_0^2)\, dt + (m_0 \to m_1) + (m_0 \to m_2) \right]$$

$$= \frac{-2\pi^2 i}{\lambda^{1/2}(p_1^2, p_2^2, q^2)} \underbrace{\left[\theta\left(t_{back}^{(q^2)} < m_0^2 < t_{forw}^{(q^2)}\right) + \theta\left(t_{back}^{(p_1^2)} < m_2^2 < t_{forw}^{(p_1^2)}\right) + \theta\left(t_{back}^{(p_2^2)} < m_1^2 < t_{forw}^{(p_2^2)}\right) \right]}_{\text{without proof, just experimental maths.}}$$

$$= \frac{-2\pi^2}{\sqrt{-\lambda(p_1^2, p_2^2, q^2)}} \begin{cases} \underline{\text{Non-zero}} \text{ if:} & \\ \lambda(p_1^2, p_2^2, q^2) < 0 & \leftarrow \text{unphysical "gap" region} \\ \& \ \det X < 0 & \leftarrow \text{Bounded by leading Landau sing.} \\ \& \ p_1^2 < (m_0 + m_1)^2 & \\ \& \ p_2^2 < (m_0 + m_2)^2 & \text{below normal threshold} \\ \& \ q^2 < (m_1 + m_2)^2 & \\ \& \ p_1^2 m_2^2 + p_2^2 m_1^2 + q^2 m_0^2 \geq (m_0 + m_1)(m_0 + m_2)(m_1 + m_2) & \leftarrow \text{above pseudo-threshold.} \end{cases}$$