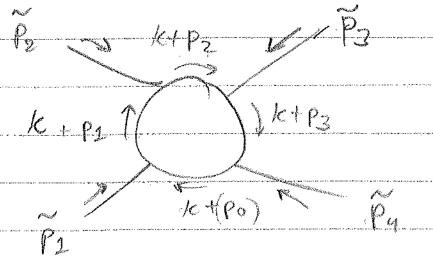
Four-point function kinematics



$$\tilde{p}_1 = p_1 - p_0$$
$$\tilde{p}_2 = p_2 - p_1$$
$$\tilde{p}_3 = p_3 - p_2$$
$$\tilde{p}_4 = p_0 - p_3$$
$$\sum_i \tilde{p}_i = 0 \quad ✓$$

$\xrightarrow[\text{take } p_0 = 0]{}$

$$p_1 = \tilde{p}_1$$
$$p_2 = \tilde{p}_2 + \tilde{p}_1$$
$$p_3 = \tilde{p}_3 + \tilde{p}_2 + \tilde{p}_1$$

(canonical gauge $p_0 = 0$)

Invariants

$$\tilde{p}_1^2 = s_1 \qquad (\tilde{p}_1+\tilde{p}_2)^2 = s_{12} = (\tilde{p}_3+\tilde{p}_4)^2$$
$$\tilde{p}_2^2 = s_2 \qquad (\tilde{p}_2+\tilde{p}_3)^2 = s_{23} = (\tilde{p}_1+\tilde{p}_4)^2$$
$$\tilde{p}_3^2 = s_3 \qquad (\tilde{p}_1+\tilde{p}_3)^2 = s_{13} = (\tilde{p}_2+\tilde{p}_4)^2$$
$$\tilde{p}_4^2 = s_4$$

$$\Rightarrow \quad \tilde{p}_i \cdot \tilde{p}_j = \frac{1}{2}(s_{ij} - s_i - s_j)$$

Conservation of momentum:

$$(\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 + \tilde{p}_4)^2 = 0$$

$$\Rightarrow \boxed{s_{12} + s_{23} + \underset{\uparrow}{s_{13}} = s_1 + s_2 + s_3 + s_4} \quad \Rightarrow \quad s_{13} = -s_{12} - s_{23} + s_1 + s_2 + s_3 + s_4$$

To be eliminated.

Connection to internal momenta

$$p_1^2 = \tilde{p}_1^2 = s_1$$
$$p_2^2 = (\tilde{p}_2+\tilde{p}_1)^2 = s_{12}$$
$$p_3^2 = (\tilde{p}_3+\tilde{p}_2+\tilde{p}_1)^2 = (-\tilde{p}_4)^2 = s_4$$
$$p_1 \cdot p_2 = \tilde{p}_1 \cdot (\tilde{p}_2+\tilde{p}_1) = \frac{1}{2}(s_{12}+s_1-s_2)$$
$$p_1 \cdot p_3 = \tilde{p}_1 \cdot (\tilde{p}_3+\tilde{p}_2+\tilde{p}_1) = -\frac{1}{2}(s_{23}-s_1-s_4)$$
$$p_2 \cdot p_3 = (\tilde{p}_2+\tilde{p}_1)\cdot(\tilde{p}_3+\tilde{p}_2+\tilde{p}_1) = \frac{1}{2}(s_{12}+s_4-s_3)$$

$$s_1 = \tilde{p}_1^2 = (p_1 - p_0)^2$$
$$s_2 = \tilde{p}_2^2 = (p_2 - p_1)^2$$
$$s_3 = \tilde{p}_3^2 = (p_3 - p_2)^2$$
$$s_4 = \tilde{p}_4^2 = (p_0 - p_3)^2$$
$$s_{12} = (\tilde{p}_1 + \tilde{p}_2)^2 = (-p_0 + p_2)^2$$
$$s_{23} = (\tilde{p}_2 + \tilde{p}_3)^2 = (-p_1 + p_3)^2$$
$$s_{13} = (\tilde{p}_1 + \tilde{p}_3)^2 = (p_1 - p_2 + p_3 - p_0)^2$$

Invarance of scalar D function.

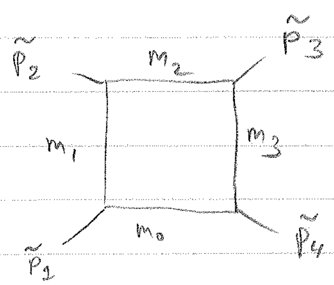
Under $p_1 \leftrightarrow p_2$, $m_1 \leftrightarrow m_2$ or $p_2 \leftrightarrow p_3$, $m_2 \leftrightarrow m_3$ or $p_1 \leftrightarrow p_3$, $m_1 \leftrightarrow m_3$ (From integral rep)

$$D_0(s_1, s_2, s_3, s_4; s_{12}, s_{23}; m_0, m_1, m_2, m_3)$$

$$= D_0(s_{12}, s_2, s_{23}, s_4; s_1, s_3; m_0, m_2, m_1, m_3) \qquad [\text{cross } \#1] \leftarrow$$

$$= D_0(s_1, s_{23}, s_3, s_{12}; s_4, s_2; m_0, m_1, m_3, m_2) \qquad [\text{cross } \#2]$$

$$= D_0(s_4, s_{23}, s_2, s_{12}; s_1, s_3; m_0, m_3, m_1, m_2)$$



rotate clockwise. $\tilde{p}_1 \to \tilde{p}_2 \to \tilde{p}_3 \to \tilde{p}_4 \to \tilde{p}_1$

$$D_0(\tilde{p}_1^2, \tilde{p}_2^2, \tilde{p}_3^2, \tilde{p}_4^2; (\tilde{p}_1+\tilde{p}_2)^2, (\tilde{p}_2+\tilde{p}_3)^2; m_0, m_1, m_2, m_3)$$

$$= D_0(\tilde{p}_2^2, \tilde{p}_3^2, \tilde{p}_4^2, \tilde{p}_1^2; (\tilde{p}_2+\tilde{p}_3)^2, (\tilde{p}_3+\tilde{p}_4)^2; m_1, m_2, m_3, m_0)$$

$$= D_0(s_2, s_3, s_4, s_1; s_{23}, s_{12}; m_1, m_2, m_3, m_0)$$

rotate counter clockwise $\tilde{p}_1 \to \tilde{p}_4 \to \tilde{p}_3 \to \tilde{p}_2 \to \tilde{p}_1$
$(\tilde{p}_4 + \tilde{p}_1)$

$$= D_0(s_4, s_1, s_2, s_3; s_{23}, s_{12}; m_3, m_0, m_1, m_2)$$

check: $D_0(s_1\ s_2\ s_3\ s_4\ s_{12}\ s_{23}\ m_0\ m_1\ m_2\ m_3)$ cross #1

$= D_0(s_{12}\ s_2\ s_{23}\ s_4\ s_1\ s_3\ m_0\ m_2\ m_1\ m_3)$ rotate clockwise

$= D_0(s_2\ s_{23}\ s_4\ s_{12}\ s_3\ s_1\ m_2\ m_1\ m_3\ m_0)$ cross #2

$= D_0(s_2\ s_1\ s_4\ s_3, s_{12}\ s_{23}, m_2\ m_1\ m_0\ m_3)$ rotatate counterclockwise twice.

$= D_0(s_4\ s_3\ s_2\ s_1, s_{12}\ s_{23}, m_0\ m_3\ m_2\ m_1)$

$\therefore$ can reverse order.

24 distinct possibilities in all.