Tensor A functions in $d=4-2\epsilon$ dimensions

$$\overbrace{I^{\mu\nu\cdots}}^{P} = \mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\,\frac{k^\mu k^\nu \cdots (P)}{k^2-m_0^2} = \frac{i e^{-\gamma_E\epsilon}}{(4\pi)^{d/2}}\,\overbrace{A^{\mu\nu\cdots}}^{P=2r}$$

Integrate via simple application of master tensor integral.
$P=2n$, only $r=n$ contributes. $N=1$ (denominators)
$Q^2=0, \quad A^2=-m_0^2 \quad \Rightarrow \quad \Delta = m_0^2$.

$$\overbrace{A^{\mu\nu\cdots}}^{2r} = \mu^{2\epsilon}\frac{(-1)^{r+1}}{2^r}\, e^{\gamma_E\epsilon}\,\Gamma\left(1-r-\frac{d}{2}\right)\left(g^{\mu\nu}g^{\cdots}\cdots+\text{perm.}\right)(m_0^2)^r\left(\frac{1}{m_0^2}\right)^{1-d/2}$$

Write $d=4-2\epsilon$

$$= \left(g^{\mu\nu\cdots}+\text{perm}\right)\mu^{2\epsilon}\frac{(-1)^{r+1}}{2^r}\,e^{\gamma_E\epsilon}\,\Gamma(-1-r+\epsilon)\,(m_0^2)^r\left(\frac{1}{m_0^2}\right)^{-1+\epsilon}$$

$$= \left(g^{\mu\nu}\cdots+\text{perm}\right)(m_0^2)^{r+1}\frac{(-1)^{r+1}}{2^r}\,\mu^{2\epsilon}e^{\gamma_E\epsilon}\,\Gamma(-1-r+\epsilon)\left(\frac{1}{m_0^2}\right)^{\epsilon}$$

Then use $e^{\gamma_E\epsilon}\,\Gamma(-n+\epsilon) \approx \frac{(-1)^n}{n!}\left(\frac{1}{\epsilon}+H_n+\mathcal{O}(\epsilon)\right)$

Harmonic number: $H_n = \sum_{k=1}^{n}\frac{1}{k}$

$$= \left(g^{\mu\nu}g^{\cdots}+\text{perm.}\right)(m_0^2)^{r+1}\frac{(-1)^{r+1}}{2^r}\,\frac{(-1)^{r+1}}{(r+1)!}\left(\frac{1}{\epsilon}+\ln\left(\frac{\mu^2}{m_0^2}\right)+H_{r+1}\right)$$

$$= \left(g^{\mu\nu}g^{\cdots}+\text{perm.}\right)\underbrace{\frac{(m_0^2)^{r+1}}{2^r(r+1)!}\left(\frac{1}{\epsilon}+\ln\left(\frac{\mu^2}{m_0^2}\right)+H_{r+1}\right)}_{A_{\underbrace{00\cdots0}_{2r}}}$$

$r+1^{st}$ Harmonic number

Useful to have $A_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n}}$

Start with

$$A^{\mu_1\cdots\mu_p}(d,\{1\}^m_{k+p}) = \underbrace{\mathcal{N}_\epsilon}_{\left(\frac{ie^{-\gamma_E\epsilon}}{(4\pi)^{d/2}}\right)^{-1}} \mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{k^{\mu_1}\cdots k^{\mu_p}}{(k+p)^2-m^2}$$

$$= \quad \text{shift: } k\to k-p$$

$$= \mathcal{N}_\epsilon\,\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{(k-p)^{\mu_1}\cdots(k-p)^{\mu_p}}{k^2-m^2}$$

$$= \mathcal{N}_\epsilon\sum_{n=0}^{p}(-1)^n\,\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{\{[p]^n[k]^{p-n}\}^{\mu_1\cdots\mu_p}}{k^2-m^2}$$

$$= \mathcal{N}_\epsilon\sum_{n=0}^{p}(-1)^n\{[p]^n[g]^{\frac{p-n}{2}}\}^{\mu_1\cdots\mu_p}A_{\underbrace{0\cdots0}_{p-n}}(m)$$

Match against general formula:

$$A^{\mu_1\cdots\mu_p}(d,\{1\}^m_{k+p}) = \sum_{n=0}^{p}\{[p]^n[g]^{\frac{p-n}{2}}\}^{\mu_1\cdots\mu_p}A_{\underbrace{0\cdots0}_{p-n}\underbrace{1\cdots1}_{n}}(m)$$

Define $2r=p-n$

$$\therefore\quad \boxed{A_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n}} = (-1)^n A_{\underbrace{0\cdots0}_{2r}}}$$

Tensor A functions for negative r to $\mathcal{O}(\epsilon)$

If $r=-1$,

$$A^{(r=-1)}(m_0) = \mu^{2\epsilon}\,\frac{(-1)^{1-1}}{2^{-1}}\; e^{\gamma_E\epsilon}\,\Gamma(-1+\epsilon+1)\left(\frac{1}{m_0^2}\right)^{2-\frac{4-2\epsilon}{2}}$$

$$= 2\,e^{\gamma_E\epsilon}\,\Gamma(\epsilon)\left(\frac{\mu^2}{m_0^2}\right)^{\epsilon}$$

$$\simeq \frac{2}{\epsilon} + 2\ln\left(\frac{\mu^2}{m_0^2}\right) + \epsilon\left[\frac{\pi^2}{6} + \ln^2\left(\frac{\mu^2}{m_0^2}\right)\right] + \ldots$$

$$A^{(r=-1)}(0) = \frac{2}{\epsilon_{UV}} - \frac{2}{\epsilon_{IR}} = 0.$$

If $r \leq -2$

$$A_{\underbrace{0\cdots0}_{2r}}(m_0) = \mu^{2\epsilon}\,\frac{(-1)^{1+r}}{2^r}\,\underbrace{e^{\gamma_E\epsilon}\,\Gamma(-1+\epsilon-r)}_{(-2-r)!\left(1+\epsilon H_{-2-r}+\ldots\right)}\left(\frac{1}{m_0^2}\right)^{\overbrace{1-\frac{4-2\epsilon}{2}-r}^{-1-r+\epsilon}}$$

$$= \frac{(-1)^{1+r}}{2^r}\,(-2-r)!\left(\frac{1}{m_0^2}\right)^{-1-r}\left(1+\epsilon H_{-2-r}+\ldots\right)\left(\frac{\mu^2}{m_0^2}\right)^{\epsilon}$$

$$= \frac{(-1)^{1+r}}{2^r}\,(-2-r)!\left(\frac{1}{m_0^2}\right)^{-1-r}\left[1+\epsilon\left(H_{-2-r}+\ln\left(\frac{\mu^2}{m_0^2}\right)\right)+\ldots\right]$$

$$A_{\underbrace{0\cdots0}_{2r}}(0) = \frac{(-1)^{1+r}}{2^r}\,(-2-r)!\left(\frac{1}{-i\varepsilon}\right)^{-1+r} \qquad \text{[Power IR divergent]}$$