Recursion relation for the reduction of B functions in $d=4-2\epsilon$ dimensions

— Denner & Dittmaier

Start with definition of $B^{\mu_1\cdots\mu_P}$ — contract with external momentum p_{μ_P}.

$$p_{\mu_P} B^{\mu_1\cdots\mu_P} = \left(\mathcal{N}_\epsilon\right) \mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{(p\cdot k)\,k^{\mu_1}\cdots k^{\mu_{P-1}}}{D_0(k)\,D_1(k)}$$

$$\begin{cases} D_0(k) = k^2 - m_0^2 \\ D_1(k) = (k+p)^2 - m_1^2 \\ f = p^2 - m_1^2 + m_0^2 \end{cases}$$

Then use: $p\cdot k = \frac{1}{2}(D_1 - D_0 - f)$

$$p_{\mu_P} B^{\mu_1\cdots\mu_P} = \left(\mathcal{N}_\epsilon\right) \mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{\frac{1}{2}\left[D_1 - D_0 - f\right]k^{\mu_1}\cdots k^{\mu_{P-1}}}{D_0(k)\,D_1(k)}$$

Expand — cancel denominators

$$p_{\mu_P} B^{\mu_1\cdots\mu_P} = \left(\mathcal{N}_\epsilon\right)\frac{1}{2}\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\left[\frac{k^{\mu_1}\cdots k^{\mu_{P-1}}}{k^2-m_0^2} - \frac{k^{\mu_1}\cdots k^{\mu_{P-1}}}{(k-p)^2-m_1^2} - f\frac{k^{\mu_1}\cdots k^{\mu_{P-1}}}{D_1(k)\,D_0(k)}\right]$$

$$= \frac{1}{2}\left[A^{\mu_1\cdots\mu_{P-1}}(d;\{1_0^{m_0}\}) - A^{\mu_1\cdots\mu_{P-1}}(d;\{1_p^{m_1}\}) - f\,B^{\mu_1\cdots\mu_{P-1}}(d;\{1,1\})\right]$$

Insert tensor decomposition into both sides:

$$2\,p_{\mu_P}\sum_{n,r}\left\{[p]^n[g]^r\right\}^{\mu_1\cdots\mu_P} B_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n}} \qquad \boxed{n = P-2r}$$

$$= \left\{[g]^{\frac{P-1}{2}}\right\}^{\mu_1\cdots\mu_{P-1}} A_{\underbrace{0\cdots0}_{\frac{P-1}{2}}}(m_0) - \sum_{n,r}\left\{[p]^n[g]^r\right\}^{\mu_1\cdots\mu_{P-1}} A_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n}}(m_1)$$

$$(-1)^n A_{\underbrace{0\cdots0}_{2r}}(m_1^2)$$

$$- f\sum_{n,r}\left\{[p]^n[g]^r\right\}^{\mu_1\cdots\mu_{P-1}} B_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n}} \qquad \boxed{n = P-2r-1}$$

LHS:

$$2p_{\mu_p}\sum_{n=0}^{p}\left\{[p]^n[g]^{\frac{p-n}{2}}\right\}^{\mu_1\cdots\mu_p} B_{\underbrace{0\cdots0}_{p-n}\underbrace{1\cdots1}_{n}}$$

$$=\sum_{n=1}^{p} 2p^2\left\{[p]^{n-1}[g]^{\frac{p-n}{2}}\right\} B_{\underbrace{0\cdots0}_{p-n}\underbrace{1\cdots1}_{n}} + \sum_{n=0}^{p} 2(n+1)\left\{[p]^{n+1}[g]^{\frac{p-n}{2}-1}\right\}^{\mu_1\cdots\mu_{p-1}} B_{\underbrace{0\cdots0}_{p-n}\underbrace{1\cdots1}_{n}}$$

$n\to n'+1$ (drop prime) $\qquad\qquad n\to n'-1$ (drop prime)

$$=\sum_{n=0}^{p-1} 2p^2\left\{[p]^{n}[g]^{\frac{p-n-1}{2}}\right\}^{\mu_1\cdots\mu_{p-1}} B_{\underbrace{0\cdots0}_{p-n-1}\underbrace{1\cdots1}_{n+1}} + \sum_{n=1}^{p+1} 2n\left\{[p]^{n}[g]^{\frac{p-n-1}{2}}\right\}^{\mu_1\cdots\mu_{p-1}} B_{\underbrace{0\cdots0}_{p-n+1}\underbrace{1\cdots1}_{n-1}}$$

Match coefficients for $n>0$ against terms in RHS.

can drop 1st term in RHS

$$2p^2 B_{\underbrace{0\cdots0}_{p-n-1}\underbrace{1\cdots1}_{n+1}} + 2n B_{\underbrace{0\cdots0}_{p-n+1}\underbrace{1\cdots1}_{n-1}} = -(-1)^n A_{\underbrace{0\cdots0}_{p-n-1}}(m_1) - f B_{\underbrace{0\cdots0}_{p-n-1}\underbrace{1\cdots1}_{n}}$$

or

$$B_{\underbrace{0\cdots0}_{p-n+1}\underbrace{1\cdots1}_{n-1}} = \frac{-1}{2n}\left[(-1)^n A_{\underbrace{0\cdots0}_{p-n-1}}(m_1) + f B_{\underbrace{0\cdots0}_{p-n-1}\underbrace{1\cdots1}_{n}} + 2p^2 B_{\underbrace{0\cdots0}_{p-n-1}\underbrace{1\cdots1}_{n+1}}\right] \qquad \begin{matrix} n\geq 1 \\ r>0\end{matrix}$$

clean up: take $n\to n+1$

$$B_{\underbrace{0\cdots0}_{p-n}\underbrace{1\cdots1}_{n}} = \frac{-1}{2(n+1)}\left[(-1)^{n+1} A_{\underbrace{0\cdots0}_{p-n-2}} + f B_{\underbrace{0\cdots0}_{p-n-2}\underbrace{1\cdots1}_{n+1}} + 2p^2 B_{\underbrace{0\cdots0}_{p-n-2}\underbrace{1\cdots1}_{n+2}}\right] \qquad \begin{matrix} n\geq 0 \\ r>0\end{matrix}$$

Write $2r = p-n$

$$B_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n}} = \frac{-1}{2(n+1)}\left[(-1)^{n+1} A_{\underbrace{0\cdots0}_{2(r-1)}}(m_1) + f B_{\underbrace{0\cdots0}_{2(r-1)}\underbrace{1\cdots1}_{n+1}} + 2p^2 B_{\underbrace{0\cdots0}_{2(r-1)}\underbrace{1\cdots1}_{n+2}}\right] \qquad \begin{matrix} n\geq 0 \\ r>0\end{matrix}$$

Obtain $B_{\underbrace{1\cdots1}_{n}}$ by direct integration.