Reduction case I: $\operatorname{Det} Z \neq 0$

Solve for $C_{\cdots}$ starting from primary reduction formulae.

Sum over k in $(1)_{LHS}$, and raise n_k by 1:

$$\sum_{k=1}^{2} S^{k}_{k\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} = \sum_{k,l=1}^{2} \underset{\substack{\uparrow\\ \text{Define: } Z_{kl}}}{p_k\cdot p_l}\, C_{kl\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} + \underbrace{\sum_{k=1}^{2}(n_k+1)}_{n_1+1+n_2+1} C_{00\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}$$

$$\sum_{k,l=1}^{2} Z_{kl}\, C_{kl\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} = -(n_1+n_2+2)\, C_{00\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} + \sum_{k=1}^{2} S^{k}_{k\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}$$

Insert into $(2)_{LHS}$.

$$S^{00}_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} = \sum_{i,j}' p_i\cdot p_j\, C_{ij\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} + \big(d+2(n_1+n_2+r)\big)\, C_{00\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}$$

$$= -(n_1+n_2+2)\, C_{000\cdots01\cdots12\cdots2} + \sum_{k=1}^{2} S^{k}_{k0\cdots01\cdots12\cdots2} + \big(d+2(n_1+n_2+r)\big)\, C_{000\cdots01\cdots12\cdots2}$$

$$= (d+n_1+n_2+2r-2)\, C_{00\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} + \sum_{k=1}^{2} S^{k}_{k\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}$$

Solve for $C_{000\cdots01\cdots12\cdots2}$

$$C_{00\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} = \frac{1}{d-2+n_1+n_2+2r}\left[S^{00}_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} - \sum_{k=1}^{2} S^{k}_{k\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} \right] \qquad \text{(A)}$$

Next, obtain $C_{i0\cdots01\cdots12\cdots2}$ using $(1)_{LHS}$

$$\sum_{l=1}^{2} Z_{kl}\, \underset{\substack{\uparrow\\ \text{isolate:}}}{C_{l\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}} = S^{k}_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} - n_k\, C_{\hat{k}00\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}$$

Multiply by $(Z^{-1})_{ik}$, sum over k.

$$C_{i\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} = \sum_{k=1}^{2} (Z^{-1})_{ik}\left[S^{k}_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} - n_k\, C_{\hat{k}00\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} \right] \qquad \text{(B)}$$

Insert $(1)_{RHS}$ & $(2)_{RHS}$ expressions for S into (A) & (B).

$\boxed{A:}$

$$C_{00\,0\cdots0\,1\cdots1\,2\cdots2} = \frac{1}{d-2+n_1+n_2+2r}\Big[B_{0\cdots0\,1\cdots1\,2\cdots2}(\hat{D}_0) + m_0^2\, C_{0\cdots0\,1\cdots1\,2\cdots2}$$
$$- \sum_{k=1}^{2} \frac{1}{2}\Big(-B_{k0\cdots0\,1\cdots1\,2\cdots2}(\hat{D}_0) - f_k\, C_{k0\cdots0\,1\cdots1\,2\cdots2}\Big)\Big]$$

apply identity.

$$= \frac{1}{d-2+n_1+n_2+2r}\Big[B_{0\cdots0\,1\cdots1\,2\cdots2}(\hat{D}_0) + m_0^2\, C_{0\cdots0\,1\cdots1\,2\cdots2}$$
$$- \frac{1}{2} B_{0\cdots0\,1\cdots1\,2\cdots2}(\hat{D}_0) + \frac{1}{2}\sum_{k=1}^{2} f_k\, C_{k0\cdots0\,1\cdots1\,2\cdots2}\Big]$$

$\therefore$

$$C_{00\,\underbrace{0\cdots0}_{2r}\,\underbrace{1\cdots1}_{n_1}\,\underbrace{2\cdots2}_{n_2}} = \frac{1}{2(d-2+n_1+n_2+2r)}\Big[B_{\underbrace{0\cdots0}_{2r}\,1\cdots1\,2\cdots2}(\hat{D}_0) + 2m_0^2\, C_{0\cdots0\,1\cdots1\,2\cdots2}$$
$$+ \sum_{k=1}^{2} f_k\, C_{k0\cdots0\,1\cdots1\,2\cdots2}\Big] \qquad \text{(A)}$$

$\boxed{B:}$

$$C_{i\,\underbrace{0\cdots0}_{2r}\,\underbrace{1\cdots1}_{n_1}\,\underbrace{2\cdots2}_{n_2}} = \sum_{k=1}^{2} (Z^{-1})_{ik}\Big[\frac{1}{2}\delta_{n_k,0}\, B_{\underbrace{0\cdots0}_{2r}\,\underbrace{1\cdots1}_{n_{\bar{k}}}}(\hat{D}_k) - \frac{1}{2} B_{\underbrace{0\cdots0}_{2r}\,\underbrace{1\cdots1}_{n_1}\,\underbrace{2\cdots2}_{n_2}}(\hat{D}_0)$$
$$- \frac{1}{2} f_k\, C_{\underbrace{0\cdots0}_{2r}\,\underbrace{1\cdots1}_{n_1}\,\underbrace{2\cdots2}_{n_2}} - n_k\, C_{\hat{k}00\,\underbrace{0\cdots0}_{2r}\,\underbrace{1\cdots1}_{n_1}\,\underbrace{2\cdots2}_{n_2}}\Big] \qquad \text{(B)}$$

$$(Z^{-1})_{ij} = \frac{1}{\mathrm{Det}\ Z}(\tilde{Z})_{ij}$$
$$= \frac{1}{\frac{1}{4}\lambda(q^2, p_1^2, p_2^2)}\begin{pmatrix} p_2^2 & -p_1\cdot p_2 \\ -p_1\cdot p_2 & p_1^2 \end{pmatrix}_{ij} \qquad p_1\cdot p_2 = -\frac{1}{2}(q^2 - p_1^2 - p_2^2)$$

and $f_k = p_k^2 - m_k^2 + m_0^2$

Mathematica implementation [CASE 1]

use invariance $C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}(\quad) = C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_2}\underbrace{2\cdots2}_{n_1}}(\quad ')$ if $n_1 < n_2$

In (A), set $n_1 = n_2 = 0$, $r \to r-1$

$$C_{\underbrace{0\cdots0}_{2r}} = \frac{1}{2(d-2+2(r-1))}\left[B_{\underbrace{0\cdots0}_{2(r-1)}}(\hat{D}_0) + 2m_0^2\, C_{\underbrace{0\cdots0}_{2(r-1)}} + \sum_{k=1}^{2} f_k\, C_{k\underbrace{0\cdots0}_{2(r-1)}}\right]$$

$$\frac{1}{2(d-2+2r-2)} = \frac{1}{2(\underbrace{d-4}_{-2\epsilon}+2r)} \approx \frac{1}{4r-4\epsilon}$$

$$C_{\underbrace{0\cdots0}_{2r}} = \frac{1}{4r}\left[B_{\underbrace{0\cdots0}_{2(r-1)}}(\hat{D}_0) + 2m_0^2\, C_{\underbrace{0\cdots0}_{2(r-1)}} + \sum_{k=1}^{2} f_k\, C_{k\underbrace{0\cdots0}_{2(r-1)}}\right] + \frac{\epsilon}{r}\, C_{\underbrace{0\cdots0}_{2r}}\Big|_{\substack{UV\\DIV}} \qquad r > 0.$$

Only $i=1$ is needed from (B)

$$C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1+1}\underbrace{2\cdots2}_{n_2}} = \frac{1}{2}\sum_k{}' (Z^{-1})_{1k}\Big[\delta_{n_k,0}\, B_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_{1(k)}}}(\hat{D}_k) - B_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}(\hat{D}_0)$$
$$- f_k\, C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} - 2n_k\, C_{\hat{k}00\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}\Big]$$

take $n_1 \to n_1 - 1$

recall: $Z^{-1} = \frac{1}{\det Z}\cdot \tilde{Z}$

$$C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} = \frac{1}{2\det Z}\sum_k{}' \tilde{Z}_{1k}\Big[\delta_{n_k-\delta_{k1},0}\, B_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_{1(k)}-\delta_{1k}}}(\hat{D}_k) - B_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1-1}\underbrace{2\cdots2}_{n_2}}(\hat{D}_0)$$
$$- f_k\, C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1-1}\underbrace{2\cdots2}_{n_2}} - 2(n_k-\delta_{k1})\, C_{\hat{k}00\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1-1}\underbrace{2\cdots2}_{n_2}}\Big] \qquad n_1 > 0$$

Passarino-Veltman reduction formulae in matrix form

In reduction case I, eqn [A] multiply by $2(d-2+n_1+n_2+2r)$ and bring $B_{0\cdots 0 1\cdots 1 2\cdots 2}(\hat{D}_0)$ to other side:

$$2m_0^2\, C_{\underbrace{0\cdots 0}_{2r}\underbrace{1\cdots 1}_{n_1}\underbrace{2\cdots 2}_{n_2}} + \sum_{k=1}^{2} f_k\, C_{k\,0\cdots 0 1\cdots 1 2\cdots 2}$$

$$= 2(d-2+n_1+n_2+2r)\, C_{00\underbrace{0\cdots 0}_{2r}\underbrace{1\cdots 1}_{n_1}\underbrace{2\cdots 2}_{n_2}} - B_{0\cdots 0 1\cdots 1 2\cdots 2}(\hat{D}_0) \qquad (*)$$

Then in reduction formula ① multiply by 2, move $f_k C_{0\cdots 0 1\cdots 1 2\cdots 2}$ and $n_k C_{\hat{k}000\cdots 0 1\cdots 1 2\cdots 2}$ to other sides

$$f_k\, C_{0\cdots 0 1\cdots 1 2\cdots 2} + \sum_{l=1}^{2} 2p_k\cdot p_l\, C_{l\,0\cdots 0 1\cdots 1 2\cdots 2}$$

$$= \delta_{n_k 0}\, B_{\underbrace{0\cdots 0}_{2r}\underbrace{1\cdots 1}_{n_{\bar{k}}}}(\hat{D}_k) - B_{\underbrace{0\cdots 0}_{2r}\underbrace{1\cdots 1}_{n_1}\underbrace{2\cdots 2}_{n_2}}(\hat{D}_0) - 2n_k\, C_{\hat{k}00\,0\cdots 0 1\cdots 1 2\cdots 2} \qquad (**)$$

Assemble (*) & (**) in matrix form.

$$\underbrace{\begin{pmatrix} 2m_0^2 & f_1 & f_2 \\ f_1 & 2p_1\cdot p_1 & 2p_1\cdot p_2 \\ f_2 & 2p_1\cdot p_2 & 2p_2\cdot p_2 \end{pmatrix}}_{X} \begin{pmatrix} C_{\underbrace{0\cdots 0}_{2r}\underbrace{1\cdots 1}_{n_1}\underbrace{2\cdots 2}_{n_2}} \\ C_{\underbrace{0\cdots 0}_{2r}\underbrace{1\cdots 1}_{n_1+1}\underbrace{2\cdots 2}_{n_2}} \\ C_{\underbrace{0\cdots 0}_{2r}\underbrace{1\cdots 1}_{n_1}\underbrace{2\cdots 2}_{n_2+1}} \end{pmatrix} = \begin{pmatrix} 2(d-2+n_1+n_2+2r)\, C_{\underbrace{0\cdots 0}_{2r+2}\underbrace{1\cdots 1}_{n_1}\underbrace{2\cdots 2}_{n_2}} - B_{0\cdots 0 1\cdots 1 2\cdots 2}(\hat{D}_0) \\ \delta_{n_1 0}\, B_{\underbrace{0\cdots 0}_{2r}\underbrace{1\cdots 1}_{n_2}}(\hat{D}_1) - B_{0\cdots 0 1\cdots 1 2\cdots 2}(\hat{D}_0) - 2n_1\, C_{\underbrace{0\cdots 0}_{2r+2}\underbrace{1\cdots 1}_{n_1-1}\underbrace{2\cdots 2}_{n_2}} \\ \delta_{n_2 0}\, B_{\underbrace{0\cdots 0}_{2r}\underbrace{1\cdots 1}_{n_1}}(\hat{D}_2) - B_{0\cdots 0 1\cdots 1 2\cdots 2}(\hat{D}_0) - 2n_2\, C_{\underbrace{0\cdots 0}_{2r+2}\underbrace{1\cdots 1}_{n_1}\underbrace{2\cdots 2}_{n_2-1}} \end{pmatrix}$$