If all elements of $Z = \begin{pmatrix} p_1^2 & p_1 p_2 \\ p_1 p_2 & p_2^2 \end{pmatrix}$ is vanishing, [CASE 5]

Then primary reduction formula ① becomes:

$$\frac{1}{2}\Big[\delta_{n_k 0} B_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_k}}(\hat{D}_k) - B_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}(\hat{D}_0) - f_k C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}\Big] = n_k C_{\hat{k}00\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} + \text{vanishing } Z \text{ terms}$$

solve for this

if $r=-1$, use →

$$f_k C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} = -2n_k C_{\hat{k}00\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} + \delta_{n_k 0} B_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_k}}(\hat{D}_k) - B_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}(\hat{D}_0) \qquad (*)$$

And primary reduction formula ② becomes:

$$\big(d+2(n_1+n_2+r)\big)\, C_{\underbrace{0\cdots0}_{2r+2}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} = B_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}(\hat{D}_0) + m_0^2\, C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} \qquad (**)$$

To implement in Mathematica,
put $r=0$ in $(*)$ & $r \to r-1$ in $(**)$

choose k with $f_k \neq 0$ →

$$\underbrace{C_{\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}}_{P} = \frac{1}{f_k}\Big[-2n_k \underbrace{C_{\hat{k}00\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}}_{P+1} + \delta_{n_k 0} B_{\underbrace{1\cdots1}_{n_k}}(\hat{D}_k) - \underbrace{B_{\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}}_{P}(\hat{D}_0)\Big] \qquad \text{if } r=0,\ n_1+n_2=P$$

$$\underbrace{C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}}_{P} = \frac{1}{d+2(n_1+n_2+r-1)}\Big[\underbrace{B_{\underbrace{0\cdots0}_{2r-2}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}}_{P-2}(\hat{D}_0) + m_0^2 \underbrace{C_{\underbrace{0\cdots0}_{2r-2}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}}_{P-2}\Big] \qquad \text{if } r\geq 1 \text{ or } P > n_1+n_2$$

$P = 2r + n_1 + n_2$

$\Rightarrow r = \dfrac{P-n_1-n_2}{2}$

$\Rightarrow d + 2\left(n_1+n_2+\dfrac{P-n_1-n_2}{2}-1\right)$

$= d + (2n_1 + 2n_2 + P - n_1 - n_2 - 2)$

$= d + (n_1 + n_2 + P - 2)$

Alert! Be sure to disable the rule

$$C_{\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} \longrightarrow C_{\underbrace{1\cdots1}_{n_2}\underbrace{2\cdots2}_{n_1}} \quad \text{for } n_2 > n_1$$

To successfully reduce this.