Reduction formulae for negative rank $r<0$

If $\det Z \neq 0$ and $\det X \neq 0$. (If $\det Z = 0$, can use Denner-Dittmaier reduction)

Start with Passarino-Veltman reduction formula in matrix form:

$$\underbrace{\begin{pmatrix} 2m_0^2 & f_1 & f_2 \\ f_1 & 2p_1\cdot p_1 & 2p_1\cdot p_2 \\ f_2 & 2p_1\cdot p_2 & 2p_2\cdot p_2 \end{pmatrix}}_{X} \begin{pmatrix} C_{0\cdots0 1\cdots1 2\cdots2} \\ C_{0\cdots0 \underbrace{1\cdots1}_{n_1+1} 2\cdots2} \\ C_{0\cdots0 1\cdots1 \underbrace{2\cdots2}_{n_2+1}} \end{pmatrix} = \begin{pmatrix} 2(d-2+n_1+n_2+2r)\, C_{\underbrace{0\cdots0}_{2r+2} 1\cdots1 2\cdots2} - B_{0\cdots0 1\cdots1 2\cdots2}(\hat{D}_0) \\ \delta_{n_2 0}\, B_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_2}}(\hat{D}_1) - B_{0\cdots0 1\cdots1 2\cdots2}(\hat{D}_0) - 2n_1\, C_{\underbrace{0\cdots0}_{2r+2}\underbrace{1\cdots1}_{n_1-1} 2\cdots2} \\ \delta_{n_2 0}\, B_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}}(\hat{D}_2) - B_{0\cdots0 1\cdots1 2\cdots2}(\hat{D}_0) - 2n_2\, C_{\underbrace{0\cdots0}_{2r+2} 1\cdots1 \underbrace{2\cdots2}_{n_2-1}} \end{pmatrix}$$

Multiply at left by $X^{-1} = \frac{1}{\det X}\tilde{X}$ and take first equation:

$$C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} = \frac{1}{\det X}\Bigg[\tilde{X}_{00}\Big(2(d-2+n_1+n_2+2r)\, C_{\underbrace{0\cdots0}_{2r+2}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} - B_{0\cdots0 1\cdots1 2\cdots2}(\hat{D}_0)\Big)$$

$$+\sum_{j=1}^{2}\tilde{X}_{0j}\Big(\delta_{n_j,0}\, B_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{\bar{j}}}(\hat{D}_j) - B_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}(\hat{D}_0) - 2n_j\, C_{\hat{j}\,\underbrace{0\cdots0}_{2r+2}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}\Big)\Bigg]$$

$$\tilde{X}_{00} = 4\det Z$$

$$\tilde{X}_{0i} = -\sum_{j=1}^{2} 2\tilde{Z}_{ij} f_j$$

If $\det X = 0$, then condition for physical threshold is satisfied.

[CASE II] If $\det X = 0$,

Multiply at left by $\tilde{X}$. LHS vanishes provided it is less singular than $\frac{1}{\det X}$.

Take first equation:

$$0 = \tilde{X}_{00}\left(2(d-2+n_1+n_2+2r)\, C_{\underbrace{0\cdots0}_{2r+2}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}} - B_{0\cdots01\cdots12\cdots2}(\hat{D}_0)\right)$$

$$+\sum_{j=1}^{2}\tilde{X}_{0j}\left(\delta_{n_j,0}\, B_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{\bar{j}}}(\hat{D}_j) - B_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}(\hat{D}_0) - 2n_j\, C_{\hat{j}\,\underbrace{0\cdots0}_{2r+2}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}\right)$$

Solve for $C_{\underbrace{0\cdots0}_{2r+2}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}$, and take $r\to r-1$ (recall $\tilde{X}_{00} = 4\det Z$)

$$C_{0\cdots01\cdots12\cdots2} = \frac{1}{2[d-2+n_1+n_2+2(r-1)]}\, B_{\underbrace{0\cdots0}_{2(r-1)}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}(\hat{D}_0)$$

$$-\frac{1}{2[d-2+n_1+n_2+2(r-1)]}\,\frac{1}{4\det Z}\sum_{j=1}^{2}\tilde{X}_{0j}\Bigg(\delta_{n_j,0}\, B_{\underbrace{0\cdots0}_{2(r-1)}\underbrace{1\cdots1}_{\bar{j}}}(\hat{D}_j)$$

$$- B_{\underbrace{0\cdots0}_{2(r-1)}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}(\hat{D}_0) - 2n_j\, C_{\hat{j}\,\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}\Bigg)$$

$$d-2+n_1+n_2+2(r-1) = \cancel{4}-2\epsilon-\cancel{2}+n_1+n_2+2r-\cancel{2}$$

$$= -2\epsilon+2r+n_1+n_2$$

If $2r = -n_1-n_2$ then RHS develops $\frac{1}{\epsilon}$ pole.

$\Rightarrow$ need $O(\epsilon)$ terms from B and C functions.

... seems useless

If $\det X = 0$, then condition for physical threshold is satisfied.

Move first column of Passarino-Veltman formula to RHS, and delete top row.

$$2\underbrace{\begin{pmatrix} p_1\cdot p_1 & p_1\cdot p_2 \\ p_1\cdot p_2 & p_2\cdot p_2 \end{pmatrix}}_{Z}\begin{pmatrix} C_{0\cdots 0 1\cdots 1 2\cdots 2} \\ C_{0\cdots 0 1\cdots 1 2\cdots 2} \end{pmatrix} = \begin{pmatrix} \sim \text{same} \sim \; - f_1 C_{0\cdots 0 \underbrace{1\cdots 1}_{n_1+1} 2\cdots 2} \\ \sim \text{same} \sim \; - f_2 C_{0\cdots 0 1\cdots 1 \underbrace{2\cdots 2}_{n_2+1}} \end{pmatrix}$$

Multiply at left by $\frac{1}{2} Z^{-1} = \frac{1}{2}\frac{\tilde{Z}}{\det Z}$.

$$C_{\hat{i}\underbrace{0\cdots 0}_{2r}\underbrace{1\cdots 1}_{n_1}\underbrace{2\cdots 2}_{n_2}} = \frac{1}{2\det Z}\sum_{j=1}^{2}\tilde{Z}_{ij}\Big(\delta_{n_j,0}\, B_{\underbrace{0\cdots 0}_{2r}\underbrace{1\cdots 1}_{n_j}}(\hat{D}_j) - B_{0\cdots 0 1\cdots 1 2\cdots 2}(\hat{D}_0) - 2n_j C_{\hat{j}\underbrace{0\cdots 0}_{2r+2} 1\cdots 1 2\cdots 2}$$

$$- \frac{1}{2\det Z}\Big(\underbrace{\sum_{j=1}^{2}\tilde{Z}_{ij} f_j}_{-\frac{1}{2}\tilde{X}_{0i}} C_{j\,0\cdots 0 1\cdots 1 2\cdots 2}\Big)$$

Choose i such that $\tilde{X}_{0i} = 0$
(not guarenteed)

$\Longrightarrow$ 2nd line drops out (hope that C doesn't have $\frac{1}{\tilde{X}_{0i}}$ singularity.)

take $n_i \to n_i - 1$

problem ?

$$C_{\underbrace{0\cdots 0}_{2r}\underbrace{1\cdots 1}_{n_1}\underbrace{2\cdots 2}_{n_2}} = \frac{1}{2\det Z}\sum_{j=1}^{2}\tilde{Z}_{ij}\Big(\delta_{n_j-\delta_{ij},0}\, B_{\underbrace{0\cdots 0}_{2r}\underbrace{1\cdots 1}_{n_j-\delta_{ij}}}(\hat{D}_j) - B_{\hat{i}\underbrace{0\cdots 0}_{2r}\underbrace{1\cdots 1}_{n_1}\underbrace{2\cdots 2}_{n_2}}(\hat{D}_0)$$

$$-2(n_j - \delta_{ij})\, C_{\hat{j}\hat{i}\underbrace{0\cdots 0}_{2r+2}\underbrace{1\cdots 1}_{n_1}\underbrace{2\cdots 2}_{n_2}}\Big)$$