Feynman-parametric representation of scalar one-loop integrals with repeated propagators

Definition:

$$T_N^0(d;\{\nu_1,\ldots,\nu_N\}) = \left(\frac{i e^{-\gamma_E \epsilon}}{(4\pi)^{d/2}}\right)^{-1} \mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{1}{D_1^{\nu_1}(k)\cdots D_N^{\nu_N}(k)}$$

Combine denominators (Feynman's trick)

$$= \left(\frac{i e^{-\gamma \epsilon}}{(4\pi)^{d/2}}\right)^{-1}\int_0^1 dx_1\cdots dx_N\,\delta\Big(1-\sum_{i=1}^N x_i\Big)\frac{\Gamma(\nu_1+\ldots+\nu_N)}{\Gamma(\nu_1)\cdots\Gamma(\nu_N)}x_1^{\nu_1-1}\cdots x_N^{\nu_N-1}$$

$$\times\,\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{1}{[x_1D_1+\ldots+x_ND_N]^{\nu_1+\ldots+\nu_N}}$$

Perform d^dk integral.

Denominator: $x_1D_1+\ldots+x_ND_N$

$$= k^2+2k\cdot Q+A^2+i\varepsilon$$

Define: $\Delta = Q^2-A^2-i\varepsilon$

$$= \left(\frac{i e^{-\gamma \epsilon}}{(4\pi)^{d/2}}\right)^{-1}\int_0^1 dx_1\cdots dx_N\,\delta\Big(1-\sum_i x_i\Big)\frac{\cancel{\Gamma(\nu_1+\ldots+\nu_N)}}{\Gamma(\nu_1)\cdots\Gamma(\nu_N)}x_1^{\nu_1-1}\cdots x_N^{\nu_N-1}$$

$$\mu^{2\epsilon}\frac{(-1)^{\nu_1+\ldots+\nu_N}\,i}{(4\pi)^{d/2}}\frac{\Gamma(\nu_1+\ldots+\nu_N-\frac{d}{2})}{\cancel{\Gamma(\nu_1+\ldots+\nu_N)}}\left(\frac{1}{\Delta}\right)^{\nu_1+\ldots+\nu_N-\frac{d}{2}}$$

$$= \mu^{2\epsilon}\frac{e^{\gamma_E\epsilon}\,\Gamma(\nu_1+\ldots+\nu_N-\frac{d}{2})}{\Gamma(\nu_1)\cdots\Gamma(\nu_N)}(-1)^{\nu_1+\ldots+\nu_N}$$

$$\times\int_0^1 dx_1\cdots dx_N\,x_1^{\nu_1-1}\cdots x_N^{\nu_N-1}\,\delta\Big(1-\sum_i x_i\Big)\left(\frac{1}{\Delta}\right)^{\nu_1+\ldots+\nu_N-\frac{d}{2}}$$

where $\Delta = \frac{1}{2}\sum_{i,j=1}^N x_ix_j\left[-(p_i-p_j)^2+m_i^2+m_j^2-2i\varepsilon\right]$

Feynman-parametric representation of scalar one-loop integrals with repeated propagators:

Definition:

$$T_N^0(\{\nu_1,\dots,\nu_N\}) = \left(\frac{ie^{-\gamma_E\epsilon}}{(4\pi)^{d/2}}\right)^{-1}\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{D_1^{\nu_1}(k)\cdots D_N^{\nu_N}(k)}$$

Combine denominators

$$= \left(\frac{ie^{-\gamma_E\epsilon}}{(4\pi)^{d/2}}\right)^{-1}\int_0^1 dx_1\cdots dx_N\,\delta(1-\textstyle\sum_i x_i)\,\frac{\Gamma(\nu_1+\dots+\nu_N)}{\Gamma(\nu_1)\cdots\Gamma(\nu_N)}\,x_1^{\nu_1-1}\cdots x_N^{\nu_N-1}\;\underbrace{\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{[x_1D_1+\dots+x_ND_N]^{\nu_1+\dots+\nu_N}}}_{\text{evaluate integral}}$$

Denominator: $x_1D_1+\dots+x_ND_N$
$= k^2+2k\cdot Q+A^2+i\varepsilon.$

Define $\Delta = Q^2-A^2-i\varepsilon$

$$= \left(\frac{ie^{-\gamma_E\epsilon}}{(4\pi)^{d/2}}\right)^{-1}\int_0^1 dx_1\cdots dx_N\,\delta(1-\textstyle\sum_i x_i)\,\frac{\cancel{\Gamma(\nu_1+\dots+\nu_N)}}{\Gamma(\nu_1)\cdots\Gamma(\nu_N)}\,x_1^{\nu_1-1}\cdots x_N^{\nu_N-1}$$

$$\times\,\mu^{2\epsilon}\,\frac{(-1)^{\nu_1+\dots+\nu_N}\,i}{(4\pi)^{d/2}}\,\frac{\Gamma(\nu_1+\dots+\nu_N-\frac{d}{2})}{\cancel{\Gamma(\nu_1+\dots+\nu_N)}}\left(\frac{1}{\Delta}\right)^{\nu_1+\dots+\nu_N-}$$

$$= \mu^{2\epsilon}\,\frac{e^{\gamma_E\epsilon}\,\Gamma(\nu_1+\dots+\nu_N-\frac{d}{2})}{\Gamma(\nu_1)\cdots\Gamma(\nu_N)}\,(-1)^{\nu_1+\dots+\nu_N}\int_0^1 dx_1\cdots dx_N\,x_N^{\nu_1-1}\cdots x_N^{\nu_N-1}\,\delta(1-\textstyle\sum_i x_i)\left(\frac{1}{\Delta}\right)^{\nu_1+\dots+\nu_N-\frac{d}{2}}$$

$$\text{where}\quad \Delta = \frac{1}{2}\sum_{i,j=1}^{N} x_ix_j\left[-(p_i-p_j)^2+m_i^2+m_j^2-2i\varepsilon\right]$$

Immediate corrolary: Davydychev's sum rule

$$\sum_{j=1}^N \nu_j\,T_N^0(d,\{\dots\nu_j+1\dots\}) = \mu^{2\epsilon}\sum_{j=1}^N \nu_j\,\frac{e^{\gamma_E\epsilon}\,\Gamma(\nu_1+\dots+1+\dots+\nu_N-\overbrace{\tfrac{d}{2})}^{-\frac{(d-2)}{2}}}{\Gamma(\nu_1)\cdots\underbrace{\Gamma(\nu_j+1)}_{\Gamma(\nu_j)}\cdots\Gamma(\nu_N)}\,(-1)^{\nu_1+\dots+1+\dots\nu_N}$$

factor out -1

$$\times\int_0^1 dx_1\cdots dx_N\,x_1^{\nu_1-1}\cdots x_j^{\nu_j}\cdots x_N^{\nu_N-1}\,\delta(\textstyle\sum x_i-1)\left(\frac{1}{\Delta}\right)^{\nu_1+\dots+\underbrace{1+\dots+\nu_N-d/2}_{-\frac{(d-2)}{2}}}$$

$$= (-1)\,\mu^{2\epsilon}\,\frac{e^{\gamma_E\epsilon}\,\Gamma(\nu_1+\dots+\nu_N-\frac{d-2}{2})}{\Gamma(\nu_1)\cdots\Gamma(\nu_N)}\,(-1)^{\nu_1+\dots+\nu_N}$$

$$\times\int_0^1 dx_1\cdots dx_N\,x_1^{\nu_1-1}\cdots x_N^{\nu_N-1}\left(\textstyle\sum_{j=1}^N x_j\right)\delta(1-\textstyle\sum_i x_i)\left(\frac{1}{\Delta}\right)^{\nu_1+\dots+\nu_N-\frac{d-2}{2}}$$

$$= -\,T_N^0(d-2,\{\nu_1,\dots,\nu_N\})$$

sum over loop integrals in which denominators for each successive term is repeated, is equal to a single loop integral in 2 less dimensions.