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Lorentz decomposition of D functions

Def$^{\underline{n}}$: $D^{\mu_1\cdots\mu_P}(p_1,p_2,p_3;m_0,m_1,m_2,m_3)$

$$= N_\epsilon\ \mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{k^{\mu_1}\cdots k^{\mu_P}}{(k^2-m_0^2)[(k+p_1)^2-m_1^2][(k+p_2)^2-m_2^2][(k+p_3)^2-m_3^2]}$$

Lorentz covariance permits a tensor decomposition in terms of
- external momenta &
- metric tensor.

$$D^\mu = p_1^\mu D_1 + p_2^\mu D_2 + p_3^\mu D_3$$

$$D^{\mu\nu} = p_1^\mu p_1^\nu D_{11} + p_1^\mu p_2^\nu D_{12} + p_2^\mu p_1^\nu D_{21} + \cdots + g^{\mu\nu}D_{00}$$

$\boxed{\text{Property 1}}$

Tensor coefficient $D^{\mu_1\cdots\mu_P}$ is a totally symmetric tensor (see def$^{\underline{n}}$)

$\Rightarrow$ coefficients are also symmetric:

$$D_{12} = D_{21} \quad (\text{and } D_{121} = D_{211} = D_{112} \text{ or } D_{001} = D_{010} = D_{100})$$

$$\therefore\ D^{\mu\nu} = p_1^\mu p_1^\nu D_{11} + (p_1^\mu p_2^\nu + p_2^\mu p_1^\nu) D_{12} + \cdots + g^{\mu\nu}D_{00}$$

$$= \{[p_1]^2\}^{\mu\nu} D_{11} + \{[p_1][p_2]\}^{\mu\nu} D_{12} + \cdots + g^{\mu\nu}D_{00}$$

$\boxed{\text{Property 2}}$ Further invariance $D_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}(s_1,s_2,s_3,s_4;s_{12},s_{23};m_0,m_1,m_2,m_3)$

Nothing unique about ordering of denominator factors

$\Rightarrow$ can exchange multiplicity numbers n_1, n_2 & n_3.

provided p_1^μ, p_2^μ & p_3^μ and m_1, m_2 & m_3 are appropriately exchanged.

$n_1 \leftrightarrow n_2$, $p_1^\mu \leftrightarrow p_2^\mu$ $\Rightarrow$ $D(p_1^2, (p_2-p_1)^2, (p_3-p_2)^2, p_3^2; p_2^2, (p_3-p_1)^2; \ldots)$

$$\longleftrightarrow D(p_2^2, (p_1-p_2)^2, (p_3-p_1)^2, p_3^2; p_1^2, (p_3-p_2)^2; \ldots)$$

$$= D_{0\cdots0\underbrace{1\cdots1}_{n_2}\underbrace{2\cdots2}_{n_1}3\cdots3}(s_{12}, s_2, s_{23}, s_4; \underline{s_1}, s_3; m_0, m_2, m_1, m_3)$$

$n_2 \leftrightarrow n_3$, $p_2^\mu \leftrightarrow p_3^\mu$ $\Rightarrow$ $D(p_1^2, (p_2-p_1)^2, (p_3-p_2)^2, p_3^2; p_2^2, (p_3-p_1)^2; \ldots)$

$$\longleftrightarrow D(p_1^2, (p_3-p_1)^2, (p_2-p_3)^2, p_2^2; p_3^2, (p_2-p_1)^2; \ldots)$$

$$= D_{0\cdots0 1\cdots1\underbrace{2\cdots2}_{n_3}\underbrace{3\cdots3}_{n_2}}(s_1, s_{23}, s_3, s_{12}; s_4, s_2; m_0, m_1, m_3, m_2)$$

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$n_1 \leftrightarrow n_3 \quad p_1^\mu \leftrightarrow p_3^\mu \quad \Rightarrow D\left(p_1^2, (p_2-p_1)^2, (p_3-p_2)^2, p_3^2; \; p_2^2, (p_3-p_1)^2; \ldots\right)$

$$\longleftrightarrow D\left(p_3^2, (p_2-p_3)^2, (p_1-p_2)^2, p_1^2; \; p_2^2, (p_3-p_1)^2; \ldots\right)$$

$$= D_{0\cdots0\,\underbrace{1\cdots1}_{n_3}\,2\cdots2\,\underbrace{3\cdots3}_{n_1}}\left(s_4, s_3, s_2, s_1; \; \underline{s_{12}}, s_{23}; \; m_0, m_3, m_2, m_1\right)$$

$n_1 \to n_2 \to n_3$ (cyclic)
$p_1^\mu \to p_2^\mu \to p_3^\mu$ (cyclic) $\quad \Rightarrow D\left(p_2^2, (p_3-p_2)^2, (p_1-p_3)^2, p_1^2; \; p_3^2, (p_1-p_2)^2; \ldots\right)$

$$= D_{0\cdots0\,\underbrace{1\cdots1}_{n_2}\,\underbrace{2\cdots2}_{n_3}\,\underbrace{3\cdots3}_{n_1}}\left(s_{12}, s_3, s_{23}, s_1; \; \underline{s_4}, s_2; \; m_0, m_2, m_3, m_1\right)$$

$n_1 \to n_3 \to n_2$ (cyclic)
$p_1^\mu \to p_3^\mu \to p_2^\mu$ (cyclic) $\quad \Rightarrow D\left(p_3^2, (p_1-p_3)^2, (p_2-p_1)^2, p_2^2; \; p_1^2, (p_2-p_3)^2; \ldots\right)$

$$= D_{0\cdots0\,\underbrace{1\cdots1}_{n_3}\,\underbrace{2\cdots2}_{n_1}\,\underbrace{3\cdots3}_{n_2}}\left(s_4, s_{23}, s_2, s_{12}; \; s_1, s_3; \; m_0, m_3, m_1, m_2\right)$$

Further invariances:

Return to Feynman Parameter integral rep.

$$D_{0\cdots01\cdots12\cdots23\cdots3} \sim \int_0^1 dx_1\, dx_2\, dx_3\, dx_4\; x_1^{n_1} x_2^{n_2} x_3^{n_3}\, \delta(1-x_1-x_2-x_3-x_4)\, \Delta^{2-\epsilon+r}$$

But integrate over x_1 (instead of x_4), putting
$x_1 = 1-x-y-z$
$x_2 = y$
$x_3 = z$
$x_4 = x$

$$= \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_1-k_1} \sum_{k_3=0}^{n_1-k_1-k_2} (-1)^{n_1} \binom{n_1}{k_1, k_2, k_3, n_1-k_1-k_2-k_3}$$

$$\times D_{\underbrace{0\cdots0}_{2r}\,\underbrace{1\cdots1}_{k_1}\,\underbrace{2\cdots2}_{n_2+k_2}\,\underbrace{3\cdots3}_{n_3+k_3}}\left(p_1^2, p_2^2, (p_3-p_2)^2, (p_3-p_1)^2; (p_2-p_1)^2, p_3^2; \ldots\right)$$

$$\parallel$$

$$\left(s_1, s_{12}, s_3, s_{23}; s_2, s_4; m_1, m_0, m_2, m_3\right)$$

✓ checked numerically with LoopTools

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Integrate over x_2, putting $x_1 = x$, $x_2 = 1-x-y-z$, $x_3 = z$, $x_4 = y$

$$= (-1)^{n_2} \sum_{k_1=0}^{n_2} \sum_{k_2=0}^{n_2-k_1} \sum_{k_3=0}^{n_2-k_1-k_2} \binom{n_2}{k_1\ k_2\ k_3\ n_2-k_1-k_2-k_3}$$

$$\times D_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1+k_1}\underbrace{2\cdots2}_{k_2}\underbrace{3\cdots3}_{n_3+k_3}}(s_2, s_1, s_4, s_3; s_{12}, s_{23}; m_2, m_1, m_0, m_3)$$

✓

Integrate over x_3, putting $x_1 = x$, $x_2 = y$, $x_3 = 1-x-y-z$, $x_4 = z$

$$= (-1)^{n_3} \sum_{k_1=0}^{n_3} \sum_{k_2=0}^{n_3-k_1} \sum_{k_3=0}^{n_3-k_1-k_2} \binom{n_3}{k_1, k_2, k_3, n_3-k_1-k_2-k_3}$$

$$\times D_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1+k_1}\underbrace{2\cdots2}_{n_2+k_2}\underbrace{3\cdots3}_{k_3}}(s_{23}, s_2, s_{12}, s_4; s_3, s_1; m_3, m_1, m_2, m_0)$$

✓

Apply invariance #1 $(n_1 \leftrightarrow n_2,\ p_1^\mu \leftrightarrow p_2^\mu)$

$$= (-1)^{n_3} \sum_{k_1=0}^{n_3} \sum_{k_2=0}^{n_3-k_1} \sum_{k_3=0}^{n_3-k_1-k_2} \binom{n_3}{k_1, k_2, k_3, n_3-k_1-k_2-k_3}$$

$$\times D_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_2+k_2}\underbrace{2\cdots2}_{n_1+k_1}\underbrace{3\cdots3}_{k_3}}(s_3, s_2, s_1, s_4; s_{23}, s_{12}; m_3, m_2, m_1, m_0)$$

✓

All checked with LoopTools