Primary Passarino-Veltman reduction formula

Reduction of D functions

$$D^{\mu_1\cdots\mu_p} = \mathcal{N}_\epsilon \; \mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{k^{\mu_1}\cdots k^{\mu_p}}{D_0 D_1 D_2 D_3} \qquad \text{[Definition]}$$

$D_0 = k^2 - m_0^2 + i\varepsilon$
$D_1 = (k+p_1)^2 - m_1^2 + i\varepsilon$
$D_2 = (k+p_2)^2 - m_2^2 + i\varepsilon$
$D_3 = (k+p_3)^2 - m_3^2 + i\varepsilon$

Contract both sides with (after raising P)

1. $(p_k)_{\mu_{p+1}}$ $\quad k = \{1,2,3\}$ $\quad$ raise $P \to P+1$
2. $g_{\mu_{p+1}\mu_{p+2}}$ $\quad$ raise $P \to P+2$

1. $$(p_k)_{\mu_{p+1}} D^{\mu_1\cdots\mu_p} = \mathcal{N}_\epsilon \times \mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{k^{\mu_1}\cdots k^{\mu_p}(k\cdot p_k)}{D_0 D_1 D_2 D_3}$$

write $p_k\cdot k = \frac{1}{2}\left[D_k - D_0 - f_k\right]$ $\qquad f_k = p_k^2 - m_k^2 + m_0^2$

$$(p_k)_{\mu_{p+1}} D^{\mu_1\cdots\mu_p} = \mathcal{N}_\epsilon \times \mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{k^{\mu_1}\cdots k^{\mu_p}\,\frac{1}{2}\left[D_k - D_0 - f_k\right]}{D_0 D_1 D_2 D_3}$$

$$= \frac{1}{2}\left[C^{\mu_1\cdots\mu_p}(\hat{D}_k) - C^{\mu_1\cdots\mu_p}(\hat{D}_0) - f_k D^{\mu_1\cdots\mu_p}\right] \qquad (1)$$

2. $$g_{\mu_{p+1}\mu_{p+2}} D^{\mu_1\cdots\mu_{p+2}} = \mathcal{N}_\epsilon \times \mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{k^{\mu_1}\cdots k^{\mu_p}(k^2)}{D_0 D_1 D_2 D_3}$$

write $k^2 = D_0 + m_0^2 - i\varepsilon$

$$g_{\mu_{p+1}\mu_{p+2}} D^{\mu_1\cdots\mu_{p+2}} = \mathcal{N}_\epsilon \times \mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{k^{\mu_1}\cdots k^{\mu_p}\left[D_0 + m_0^2 - i\varepsilon\right]}{D_0 D_1 D_2 D_3}$$

$$= C^{\mu_1\cdots\mu_p}(\hat{D}_0) + (m_0^2 - i\varepsilon) D^{\mu_1\cdots\mu_p} \qquad (2)$$

Let $S_k^{\mu_1\cdots\mu_p}$ and $S_{00}^{\mu_1\cdots\mu_p}$ represent both sides of (1) & (2).
Their covariant decompositions are:

$$S_k^{\mu_1\cdots\mu_p} = \sum_{r=0}^{\lfloor\frac{p}{2}\rfloor}\sum_{n_1=0}^{p-2r}\sum_{n_2=0}^{p-2r-n_1}\left\{[p_1]^{n_1}[p_2]^{n_2}[p_3]^{p-2r-n_1-n_2}[g]^r\right\}^{\mu_1\cdots\mu_p} S^k_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{p-2r-n_1-n_2}} \qquad (1a)$$

$$S_{00}^{\mu_1\cdots\mu_p} = \sum_{r=0}^{\lfloor\frac{p}{2}\rfloor}\sum_{n_1=0}^{p-2r}\sum_{n_2=0}^{p-2r-n_1}\left\{[p_1]^{n_1}[p_2]^{n_2}[p_3]^{p-2r-n_1-n_2}[g]^r\right\}^{\mu_1\cdots\mu_p} S^{00}_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{p-2r-n_1-n_2}} \qquad (2a)$$

Next, insert covariant decompositions of $D^{\mu_1\cdots\mu_p}$ into LHS of (1) & (2),

$$D^{\mu_1\cdots\mu_p} = \sum_{r=0}^{\lfloor\frac{p}{2}\rfloor}\sum_{n_1=0}^{p-2r}\sum_{n_2=0}^{p-2r-n_1}\left\{[p_1]^{n_1}[p_2]^{n_2}[p_3]^{p-2r-n_1-n_2}[g]^r\right\}^{\mu_1\cdots\mu_p} D_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{p-2r-n_1-n_2}}$$

and match with (1a) to obtain LHS rep. of $S^{k}_{0\cdots0\,1\cdots1\,2\cdots2\,3\cdots3}$.
" " " (2a) " " LHS " " $S^{00}_{0\cdots0\,1\cdots1\,2\cdots2\,3\cdots3}$.

Result

$$(1)_{LHS} = \sum_{r=0}^{p/2}\sum_{n_1=-1}^{p-1-2r}\sum_{n_2=0}^{p-2r-n_1-1}(p_k\cdot p_1)\left\{[p_1]^{n_1}[p_2]^{n_2}[p_3]^{p-2r-n_1-n_2}[g]^r\right\}^{\mu_1\cdots\mu_p} D_{1\,0\cdots0\,1\cdots1\,2\cdots2\,3\cdots3}$$

$$+\sum_{r=0}^{p/2}\sum_{n_1=0}^{p-2r}\sum_{n_2=-1}^{p-2r-n_1-1}(p_k\cdot p_2)\left\{[p_1]^{n_1}[p_2]^{n_2}[p_3]^{p-2r-n_1-n_2}[g]^r\right\}^{\mu_1\cdots\mu_p} D_{2\,0\cdots0\,1\cdots1\,2\cdots2\,3\cdots3}$$

$$+\sum_{r=0}^{p/2}\sum_{n_1=0}^{p-2r}\sum_{n_2=0}^{p-2r-n_1}(p_k\cdot p_3)\left\{[p_1]^{n_1}[p_2]^{n_2}[p_3]^{p-2r-n_1-n_2}[g]^r\right\}^{\mu_1\cdots\mu_p} D_{3\,0\cdots0\,1\cdots1\,2\cdots2\,3\cdots3}$$

$$+\sum_{r=-1}^{p/2-1}\sum_{n_1=0}^{p-2r}\sum_{n_2=0}^{p-2r-n_1}(n_k-1+1)\left\{[p_1]^{n_1}[p_2]^{n_2}[p_3]^{p-2r-n_1-n_2}[g]^r\right\}^{\mu_1\cdots\mu_p} D_{\hat{k}\,00\,0\cdots0\,1\cdots1\,2\cdots2\,3\cdots3}$$

$\therefore$

$$S^{k}_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{p-2r-n_1-n_2}} = p_k\cdot p_1\, D_{1\,0\cdots0\,1\cdots1\,2\cdots2\,3\cdots3} + p_k\cdot p_2\, D_{2\,0\cdots0\,1\cdots1\,2\cdots2\,3\cdots3}$$

$$+ p_k\cdot p_3\, D_{3\,0\cdots0\,1\cdots1\,2\cdots2\,3\cdots3} + n_k\, D_{\hat{k}\,00\,1\cdots1\,2\cdots2\,3\cdots3}$$

Put $p-2r-n_1-n_2 = n_3$

$$S^{k}_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}} = \sum_{\ell=1}^{3} p_k\cdot p_\ell\, D_{\ell\,\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}} + n_k\, D_{\hat{k}\,00\,\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}$$

Similarly,

$$(2)_{LHS} = g_{\mu_{p+1}\mu_{p+2}}D^{\mu_1\cdots\mu_{p+2}}$$

$$= g_{\mu_{p+1}\mu_{p+2}}\sum_{n_1 n_2 n_3}\left\{[p_1]^{n_1}[p_2]^{n_2}[p_3]^{n_3}[g]^{\frac{p+2-n_1-n_2-n_3}{2}}\right\}^{\mu_1\cdots\mu_{p+2}} D_{\underbrace{0\cdots0}_{p+2-n_1-n_2-n_3}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}$$

Apply contraction formula

$$(2)_{LHS} = \sum_{n_1 n_2 n_3} \Bigg[\sum_{i,j} p_i \cdot p_j \left\{ [\hat{p}_i][\hat{p}_j][p_1]^{n_1}[p_2]^{n_2}[p_3]^{n_3}[g]^{\frac{p+2-n_1-n_2-n_3}{2}} \right\}^{\mu_1\cdots\mu_p} D_{\underbrace{0\cdots0}_{p+2-n_1-n_2-n_3}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}$$

$$+ (d+p+2-2+n_1+n_2+n_3)\left\{[p_1]^{n_1}[p_2]^{n_2}[p_3]^{n_3}[g]^{\frac{p-n_1-n_2-n_3}{2}}\right\}^{\mu_1\cdots\mu_p} D_{0\cdots0\,1\cdots1\,2\cdots2\,3\cdots3} \Bigg]$$

In first line, shift $n_i \to n_i + 1$, $n_j \to n_j + 1$ (also take $p \to p-2$)

In second line, nothing.

$$(2)_{LHS} = \sum_{n_1 n_2 n_3} \Bigg[\sum_{i,j} p_i \cdot p_j \left\{ [p_1]^{n_1}[p_2]^{n_2}[p_3]^{n_3}[g]^{\frac{p-n_1-n_2-n_3}{2}} \right\}^{\mu_1\cdots\mu_p} D_{ij\,\underbrace{0\cdots0}_{p-n_1-n_2-n_3}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}$$

$$+ (d+p+n_1+n_2+n_3)\left\{[p_1]^{n_1}[p_2]^{n_2}[p_3]^{n_3}[g]^{\frac{p-n_1-n_2-n_3}{2}}\right\}^{\mu_1\cdots\mu_p} D_{\underbrace{0\cdots0}_{p+2-n_1-n_2-n_3}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}} \Bigg]$$

Match with $S^{00}_{0\cdots0\,1\cdots1\,2\cdots2\,3\cdots3}$

$$S^{00}_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}} = \sum_{i,j} p_i \cdot p_j \, D_{ij\,\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}} + (d+2(r+n_1+n_2+n_3))\, D_{00\,\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}$$

Now insert covariant decomposition of D & C into RHS of (1) & (2).

$$(1)_{RHS} = \frac{1}{2}\left[C^{\mu_1\cdots\mu_p}(\hat{D}_k) - C^{\mu_1\cdots\mu_p}(\hat{D}_0) - f_k D^{\mu_1\cdots\mu_p} \right]$$

$$= \frac{1}{2}\Bigg[\sum_{r=0}^{\lfloor p/2 \rfloor} \sum_{n_{1(k)}=0}^{p-2r} \left\{ [p_{1(k)}]^{n_{1(k)}}[p_{2(k)}]^{n_{2(k)}}[g]^r \right\}^{\mu_1\cdots\mu_p} C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_{1(k)}}\underbrace{2\cdots2}_{n_{2(k)}}}(\hat{D}_k)$$

$$- \sum_{r=0}^{\lfloor p/2 \rfloor} \sum_{n_1=0} \sum_{n_2=0} \left\{ [p_1]^{n_1}[p_2]^{n_2}[p_3]^{n_3}[g]^r \right\}^{\mu_1\cdots\mu_p} C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}(\hat{D}_0)$$

$$- f_k \sum_{r=0}^{\lfloor p/2 \rfloor} \sum_{n_1=0} \sum_{n_2=0} \left\{ [p_1]^{n_1}[p_2]^{n_2}[p_3]^{n_3}[g]^r \right\}^{\mu_1\cdots\mu_p} D_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}} \Bigg]$$

First line will match with $S^{k}_{0\cdots0\,1\cdots1\,2\cdots2\,3\cdots3}$ only if $n_k = 0$

$$S^{k}_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}} = \frac{1}{2}\left[\delta_{n_k 0}\, C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_{1(k)}}\underbrace{2\cdots2}_{n_{2(k)}}}(\hat{D}_k) - C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}(\hat{D}_0) - f_k\, D_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}} \right]$$

Similarly

$$(2)_{RHS} = C^{\mu_1\cdots\mu_p}(\hat{D}_0) + m_0^2\, D^{\mu_1\cdots\mu_p}$$

$$= \sum_{r=0}^{[p/2]}{}' \sum_{n_1}{}' \sum_{n_2}{}' \left\{[p_1]^{n_1}[p_2]^{n_2}[p_3]^{n_3}[g]^r\right\}^{\mu_1\cdots\mu_p}\Big(C_{0\cdots0 1\cdots1 2\cdots2 3\cdots3}(\hat{D}_0)$$

$$+ m_0^2 D_{0\cdots0 1\cdots1 2\cdots2 3\cdots3}\Big)$$

Match:

$$S^{00}_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}} = C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}(\hat{D}_0) + m_0^2 D_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}$$

Primary Passarino-Veltman reduction formulae

1 $$S^{k}_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}} = \sum_{l=1}^{3} p_k\cdot p_l\, D_{l\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}} + n_k D_{\hat{k}00\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}$$

$$= \frac{1}{2}\Big[\delta_{n_k 0}\, C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_{1(k)}}\underbrace{2\cdots2}_{n_{2(k)}}}(\hat{D}_k) - C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}(\hat{D}_0) - f_k D_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}\Big]$$

2 $$S^{00}_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}} = \sum_{i,j} p_i\cdot p_j\, D_{ij\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}} + \big(d+2(r+n_1+n_2+n_3)\big) D_{00\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}$$

$$= C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}(\hat{D}_0) + m_0^2 D_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}$$