Covariant decomposition of pinched C functions (valid also for non-unit weights)

$$C^{\mu_1\cdots\mu_P}(\hat{D}_k) = \sum_{r=0}^{\lfloor P/2\rfloor} \sum_{n_{1(k)}=0}^{P-2r} \left\{[p_{1(k)}]^{n_{1(k)}} [p_{2(k)}]^{n_{2(k)}} [g]^r\right\}^{\mu_1\cdots\mu_P} C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_{1(k)}}\underbrace{2\cdots2}_{n_{2(k)}}}(\hat{D}_k)$$

where $n_{i(k)} = \begin{cases} n_i & \text{if } i<k \\ n_{i+1} & \text{if } i\geq k \end{cases}$ $\qquad k=\{1,2,3\}$.

Arguments:

$C(\hat{D}_1)$: $\frac{1}{D_0 D_2 D_3}$ $\quad (\underset{s_{12}}{p_2^2}, \underset{s_3}{(p_3-p_2)^2}, \underset{s_4}{p_3^2};\ m_0, m_2, m_3)$

$C(\hat{D}_2)$: $\frac{1}{D_0 D_1 D_3}$ $\quad (\underset{s_1}{p_1^2}, \underset{s_{23}}{(p_3-p_1)^2}, \underset{s_4}{p_3^2};\ m_0, m_1, m_3)$

$C(\hat{D}_3)$: $\frac{1}{D_0 D_1 D_2}$ $\quad (\underset{s_1}{p_1^2}, \underset{s_2}{(p_2-p_1)^2}, \underset{s_{12}}{p_2^2};\ m_0, m_1, m_2)$ $\qquad$ [Normal C function]

Pinched $\hat{D}_0$:

$$C^{\mu_1\cdots\mu_P}(\hat{D}_0) = \sum_{r=0}^{\lfloor P/2\rfloor} \sum_{n_1=0} \sum_{n_2=0} \left\{[p_1]^{n_1}[p_2]^{n_2}[p_3]^{n_3}[g]^r\right\}^{\mu_1\cdots\mu_P} C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}(\hat{D}_0)$$

Arguments:
Derived by shifting $k\to k-p_1$ so that denominator $=(k^2-m_1^2)((k+p_2-p_1)^2-m_2^2)((k+p_3-p_1)^2-m_3^2)$

$$\therefore\ C(\hat{D}_0) = C(\underset{s_2}{(p_2-p_1)^2}, \underset{s_3}{(p_3-p_2)^2}, \underset{s_{23}}{(p_3-p_1)^2};\ m_1, m_2, m_3)$$

$$\tilde{C}_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}(\hat{D}_0) = (-1)^{n_1}\sum_{j=0}^{n_1}\sum_{k=0}^{n_1-j}\frac{n_1!}{j!\,k!\,(n_1-j-k)!}\, C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_2+k}\underbrace{2\cdots2}_{n_1+n_3-j-k}}(\hat{D}_0)$$

Invariances of $\tilde{C}$:

$n_1\longleftrightarrow n_2 \quad\Rightarrow\quad p_1^\mu\leftrightarrow p_2^\mu \quad\longrightarrow\quad C((p_1-p_2)^2, (p_3-p_1)^2, (p_3-p_2)^2;\ m_2, m_1, m_3)$
$\qquad = C(s_2, s_{23}, s_3;\ m_2, m_1, m_3)$

$n_2\longleftrightarrow n_3 \quad\Rightarrow\quad p_2\leftrightarrow p_3 \quad\Rightarrow\quad C((p_3-p_1)^2, (p_2-p_3)^2, (p_2-p_1)^2;\ m_1, m_3, m_2)$
$\qquad C(s_{23}, s_3, s_2;\ m_1, m_3, m_2)$

$n_1\longleftrightarrow n_3 \quad\Rightarrow\quad p_1\leftrightarrow p_3 \quad\longrightarrow\quad C((p_2-p_3)^2, (p_1-p_2)^2, (p_1-p_3)^2;\ m_3, m_2, m_1)$
$\qquad C(s_3, s_2, s_{23};\ m_3, m_2, m_1)$