Reduction case II $\det Z = 0$ $\tilde{X}_{0i} \neq 0$.

Multiply primary equation ① by $\tilde{Z}_{jk}$, and sum over k.

$$\sum_{l=1}^{3}\sum_{k=1}^{3} \underbrace{\tilde{Z}_{jk} Z_{kl}}_{=\det Z \to 0} D_{l0\cdots01\cdots12\cdots23\cdots3} + \sum_{k=1}^{3} \tilde{Z}_{jk}\, n_k\, D_{\hat{k}00\,0\cdots01\cdots12\cdots23\cdots3}$$

$$= \frac{1}{2}\sum_{k=1}^{3} \tilde{Z}_{jk}\Big(\delta_{n_k 0}\, C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1(k)}\underbrace{2\cdots2}_{n_2(k)}}(\hat{D}_k) - C_{0\cdots01\cdots12\cdots23\cdots3}(\hat{D}_0)\Big) - \frac{1}{2}\sum_{k=1}^{3}\tilde{Z}_{jk} f_k\, D_{0\cdots01\cdots12\cdots23\cdots3}$$

$\uparrow$ solve.

$$\Big(\sum_{k=1}^{3} \tilde{Z}_{jk} f_k\Big) D_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}$$

$$= \sum_{k=1}^{3}\tilde{Z}_{jk}\Big[\delta_{n_k 0}\, C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1(k)}\underbrace{2\cdots2}_{n_2(k)}}(\hat{D}_k) - C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}(\hat{D}_0) - 2n_k\, D_{\hat{k}00\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}\Big] \qquad [1]$$

Need second equation.

Start with

$$\det Z\; D_{kl\,0\cdots01\cdots12\cdots23\cdots3} = \sum_{i,j=1}^{3} \underbrace{\det Z\, \delta_{ki}\delta_{lj}}\, D_{ij\,0\cdots01\cdots12\cdots23\cdots3} \qquad \Big[\text{sum over } i, j \text{ to verify}\Big]$$

$$= \sum_{i,j=1}^{3}\Big(\tilde{Z}_{kl} Z_{ij} + \sum_{n,m=1}^{3} \tilde{\tilde{Z}}_{(kn)(lm)}\, \underbrace{Z_{nj} Z_{im}}\Big) D_{ij\,0\cdots01\cdots12\cdots23\cdots3} \qquad (*)$$

PV reduction ②

PV reduction ①

In first term, apply PV reduction ②

$$\sum_{i,j=1}^{3} Z_{ij} D_{ij\boxed{}} = C_{\boxed{}}(\hat{D}_0) + m_0^2 D_{\boxed{}} - \big(d + 2(r + n_1 + n_2 + n_3)\big) D_{00\boxed{}} \qquad (**)$$

In second term, apply PV reduction ① [will raise n_j by one later]

$$\sum_{i=1}^{2} Z_{im} D_{i\boxed{}} = \frac{1}{2}\Big[\delta_{n_m 0}\, C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1(m)}\underbrace{2\cdots2}_{n_2(m)}}(\hat{D}_m) - C_{\boxed{}}(\hat{D}_0) - f_m D_{\boxed{}}\Big] - n_m D_{\hat{m}00\boxed{}}$$

raise n_j by one.

abbr: $C_{j\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_{1(m)}}\underbrace{2\cdots2}_{n_{2(m)}}}(\hat{D}_m)$

$$\sum_{i=1}^{2} Z_{im} D_{ij\boxed{}} = \frac{1}{2}\Big[\delta_{n_m+\delta_{mj},0}\,\boxed{C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_{1(m)}+\delta_{1(m),j}}\underbrace{2\cdots2}_{n_{2(m)}+\delta_{2(m),j}}}(\hat{D}_m)} - C_{j\boxed{}}(\hat{D}_0)$$

$$- f_m D_{j\boxed{}} - 2(n_m+\delta_{mj}) D_{\hat{m}00j\boxed{}}\Big]$$

Pull this term out.

Multiply by Z_{nj} and sum over j.

$$\sum_{i,j=1}^{3} Z_{nj} Z_{im} D_{ij\boxed{}} = \frac{1}{2}\sum_{j=1}^{3} Z_{nj}\Big[\delta_{n_m+\delta_{mj},0}\, C_{j\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_{1(m)}}\underbrace{2\cdots2}_{n_{2(m)}}}(\hat{D}_m) - C_{j\boxed{}}(\hat{D}_0)$$

$$- 2(n_m+\delta_{mj}) D_{\hat{m}00j\boxed{}}\Big] - \frac{1}{2} f_m \underbrace{\sum_{j=1}^{3} Z_{nj} D_{j\boxed{}}}_{\text{Apply PV formula ① again.}}$$

$$= \frac{1}{2}\sum_{j=1}^{3} Z_{nj}\Big[\delta_{n_m+\delta_{mj},0}\, C_{j\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_{1(m)}}\underbrace{2\cdots2}_{n_{2(m)}}}(\hat{D}_m) - C_{j\boxed{}}(\hat{D}_0) - 2(n_m+\delta_{mj}) D_{\hat{m}00j\boxed{}}\Big]$$

$$-\frac{1}{2} f_m \frac{1}{2}\Big[\delta_{n_n 0}\, C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_{1(n)}}\underbrace{2\cdots2}_{n_{2(n)}}}(\hat{D}_n) - C_{\boxed{}}(\hat{D}_0) - f_n D_{\boxed{}} - 2n_n D_{\hat{n}00\boxed{}}\Big] \qquad (\ast\ast\ast)$$

Insert $(\ast\ast)$ & $(\ast\ast\ast)$ into $(\ast)$.

$$\det Z\, D_{kl0\cdots01\cdots12\cdots23\cdots3} = \tilde{Z}_{kl}\big(C_{0\cdots01\cdots12\cdots23\cdots3}(\hat{D}_0) + m_0^2 D_{0\cdots01\cdots12\cdots23\cdots3}\big)$$

$$- \tilde{Z}_{kl}\big(d+2(r+n_1+n_2+n_3)\big) D_{000\cdots01\cdots12\cdots23\cdots3}$$

$$+ \sum_{n,m=1}^{3} \tilde{Z}_{(kn)(lm)} \times \frac{1}{2}\Big\{\sum_{j=1}^{3} Z_{nj}\Big[\delta_{n_m+\delta_{mj},0}\, C_{j\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_{1(m)}}\underbrace{2\cdots2}_{n_{2(m)}}}(\hat{D}_m) - C_{j0\cdots01\cdots12\cdots23\cdots3}(\hat{D}_0)$$

$$- 2(n_m+\delta_{mj}) D_{\hat{m}00j0\cdots01\cdots12\cdots23\cdots3}\Big]$$

solve for these.

$$+ \frac{1}{2} f_m\Big[-\delta_{n_n 0}\, C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_{1(n)}}\underbrace{2\cdots2}_{n_{2(n)}}}(\hat{D}_n) + C_{0\cdots01\cdots12\cdots23\cdots3}(\hat{D}_0)$$

$$+ f_n D_{0\cdots01\cdots12\cdots23\cdots3} + 2n_n D_{\hat{n}000\cdots01\cdots12\cdots23\cdots3}\Big]\Big\}$$

$$\tilde{Z}_{kl}\,(d+2(r+n_1+n_2+n_3))\,D_{00\,0\cdots0\,1\cdots1\,2\cdots2\,3\cdots3}+\sum_{n,m=1}^{3}\tilde{\tilde{Z}}_{(kn)(lm)}\sum_{j=1}^{3}Z_{nj}\,(n_m+\delta_{mj})\,D_{\hat{m}j\,000\cdots0\,1\cdots1\,2\cdots2\,3\cdots}$$

$$=-\det Z\,D_{kl\,0\cdots0\,1\cdots1\,2\cdots2\,3\cdots3}+\tilde{Z}_{kl}\left(C_{0\cdots0\,1\cdots1\,2\cdots2\,3\cdots3}(\hat{D}_0)+m_0^2\,D_{0\cdots0\,1\cdots1\,2\cdots2\,3\cdots3}\right)$$

$$+\sum_{n,m=1}^{3}\tilde{\tilde{Z}}_{(kn)(lm)}\,\frac{1}{2}\left\{\sum_{j=1}^{3}Z_{nj}\left[\delta_{n_m+\delta_{mj},0}\;C_{j\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1(m)}\underbrace{2\cdots2}_{n_2(m)}}(\hat{D}_m)-C_{j0\cdots0\,1\cdots1\,2\cdots2\,3\cdots3}(\hat{D}_0)\right]\right.$$

$$+\frac{1}{2}f_m\left[-\delta_{n_n,0}\;C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1(n)}\underbrace{2\cdots2}_{n_2(n)}}(\hat{D}_n)+C_{0\cdots0\,1\cdots1\,2\cdots2\,3\cdots3}(\hat{D}_0)\right.$$

$$\left.\left.+f_n\,D_{0\cdots0\,1\cdots1\,2\cdots2\,3\cdots3}+2n_n\,D_{\hat{n}00\,0\cdots0\,1\cdots1\,2\cdots2\,3\cdots3}\right]\right\}$$

To implement in Mathematica, put $n_1=n_2=n_3=0$, $r\to r-1$

2nd term LHS:

$$\rightarrow\sum_{n,m=1}^{3}\tilde{\tilde{Z}}_{(kn)(lm)}\sum_{j=1}^{3}Z_{nj}\,(0+\delta_{mj})\,D_{\hat{m}j\,0\cdots0\,1\cdots1\,2\cdots2\,3\cdots3}$$

↑cancel

sum over j, $j=m$

$$=\underbrace{\sum_{n,m=1}^{3}\tilde{\tilde{Z}}_{(kn)(lm)}\,Z_{nm}}\,D_{0\cdots0\,1\cdots1\,2\cdots2\,3\cdots3}$$

$$=-2\,\tilde{Z}_{kl}\,D_{0\cdots0\,1\cdots1\,2\cdots2\,3\cdots3}.$$

1 middle term RHS:

$$\underbrace{\delta_{0+\delta_{mj},0}}_{(1-\delta_{mj})}\;C_{\underbrace{0\cdots0}_{2(r-1)}\underbrace{1\cdots1}_{\delta_{1(m),j}}\underbrace{2\cdots2}_{\delta_{2(m),j}}}(\hat{D}_m)$$

$$\therefore\;(d+\overset{r\to r-1}{2r-2}-2)\,\tilde{Z}_{kl}\,D_{\underbrace{0\cdots0}_{2r}}=\tilde{Z}_{kl}\left(C_{\underbrace{0\cdots0}_{2(r-1)}}(\hat{D}_0)+m_0^2\,D_{\underbrace{0\cdots0}_{2(r-1)}}\right)$$

$$+\frac{1}{2}\sum_{n,m=1}^{3}\tilde{\tilde{Z}}_{(kn)(lm)}\left\{\sum_{j=1}^{3}Z_{nj}\left[(1-\delta_{mj})\,C_{\underbrace{0\cdots0}_{2(r-1)}\underbrace{1\cdots1}_{\delta_{1(m),j}}\underbrace{2\cdots2}_{\delta_{2(m),j}}}(\hat{D}_m)-C_{j\underbrace{0\cdots0}_{2(r-1)}}(\hat{D}_0)\right]\right.$$

$$\left.+\frac{1}{2}f_m\left[-C_{\underbrace{0\cdots0}_{2(r-1)}}(\hat{D}_n)+C_{\underbrace{0\cdots0}_{2(r-1)}}(\hat{D}_0)+f_n\,D_{\underbrace{0\cdots0}_{2(r-1)}}\right]\right\}\qquad r>0$$

$$d+2r-4=-2\epsilon+2r$$

$$2r\,\tilde{Z}_{kl}\,D_{\underbrace{0\cdots0}_{2r}}-2\epsilon\,\tilde{Z}_{kl}\,D_{\underbrace{0\cdots0}_{2r}}\Big|_{\substack{\text{UV}\\\text{div}}}$$

$$\boxed{\begin{aligned} \underbrace{D_{0\cdots0}}_{2r} = {} & \frac{1}{2r}\Big(\underbrace{C_{0\cdots0}}_{2(r-1)}(\hat{D}_0) + m_0^2 \underbrace{D_{0\cdots0}}_{2(r-1)}\Big) + \frac{\epsilon}{r}\underbrace{D_{0\cdots0}}_{2r}\Big|_{\substack{uv\\div}} \\ & + \frac{1}{2(2r)}\frac{1}{\tilde{Z}_{kl}}\sum_{n,m=1}^{3}\tilde{\tilde{Z}}_{(kn)(lm)}\Bigg\{\sum_{j=1}^{3} Z_{nj}\Big[(1-\delta_{mj})\underbrace{C_{0\cdots0}}_{2(r-1)}\underbrace{1\cdots1}_{\delta_{1(m),j}}\underbrace{2\cdots2}_{\delta_{2(m),j}}(\hat{D}_m) - C_{j\underbrace{0\cdots0}_{2(r-1)}}(\hat{D}_0)\Big] \\ & + \frac{1}{2}f_m\Big[-\underbrace{C_{0\cdots0}}_{2(r-1)}(\hat{D}_n) + \underbrace{C_{0\cdots0}}_{2(r-1)}(\hat{D}_0) + f_n\underbrace{D_{0\cdots0}}_{2(r-1)}\Big]\Bigg\}, \quad r>0 \qquad [2] \end{aligned}}$$