Case III, $\det Z = 0$, $\tilde{X}_{0j} = 0$

but at least one of $\tilde{Z}_{ij} \neq 0$ and one of $\tilde{X}_{ij} \neq 0$.

In [1] of $\det Z = 0$ (case II),

LHS vanishes. $\Rightarrow$ solve for $\sum_{k=1}^{3} \tilde{Z}_{jk}\, 2n_k\, D_{\hat{k}00\,0\cdots 01\cdots 12\cdots 23\cdots 3}$. (take $r \to r-1$)

$$2\sum_{k=1}^{3} \tilde{Z}_{jk}\, n_k\, D_{\hat{k}\,\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}$$

$$= \sum_{l=1}^{3} \tilde{Z}_{jl}\left[\delta_{n_l 0}\, C_{\underbrace{0\cdots0}_{2(r-1)}\underbrace{1\cdots1}_{n_{1(l)}}\underbrace{2\cdots2}_{n_{2(l)}}}(\hat{D}_l) - C_{\underbrace{0\cdots0}_{2(r-1)}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}(\hat{D}_0)\right]$$

open sum in LHS:

$$2\tilde{Z}_{j1}\, n_1\, D_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1-1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}} + 2\tilde{Z}_{j2}\, n_2\, D_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2-1}\underbrace{3\cdots3}_{n_3}} + 2\tilde{Z}_{j3}\, n_3\, D_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3-1}}$$

$$= \sum_{l=1}^{3}\tilde{Z}_{jl}\left[\cdots\right]$$

Choose $(j,k)$ element of $\tilde{Z}$ that is non-vanishing

Take $n_k \to n_k + 1$, and solve for $D_{0\cdots 01\cdots 12\cdots 23\cdots 3}$

$$D_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}} = \frac{1}{2(n_k+1)\tilde{Z}_{jk}}\left[-2\sum_{l\neq k}\tilde{Z}_{jl}\, n_l\, D_{k\hat{l}\,\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}\right. \qquad [1]$$

$$\left. + \sum_{l=1}^{3}\tilde{Z}_{jl}\left(\delta_{n_l + \delta_{kl}}\, C_{\underbrace{0\cdots0}_{2(r-1)}\underbrace{1\cdots1}_{n_{1(l)}+\delta_{1(l),k}}\underbrace{2\cdots2}_{n_{2(l)}+\delta_{2(l),k}}}(\hat{D}_l) - C_{\underbrace{0\cdots0}_{2(r-1)}k\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}(\hat{D}_0)\right)\right]$$

valid for $r \geq 1$.

Note: $n_l$ is reduced by one with each iteration (because $l \neq k$)

$\Rightarrow$ Eventually, there will be no $D$'s in RHS.

If $\det X \neq 0$, then it works for all $r$.

If $r=0$, would need $C_{\underbrace{0\cdots0}_{2(r=-1)}1\cdots 1\, 2\cdots 2}$

To get second formula with $r=0$,

Start with matrix form of P.V. reduction formula, and bring first column to RHS.

$$\begin{pmatrix} f_1 & f_2 & f_3 \\ 2p_1\cdot p_1 & 2p_1\cdot p_2 & 2p_1\cdot p_3 \\ 2p_2\cdot p_1 & 2p_2\cdot p_2 & 2p_2\cdot p_3 \\ 2p_3\cdot p_1 & 2p_3\cdot p_2 & 2p_3\cdot p_3 \end{pmatrix} \begin{pmatrix} D_{0\cdots0\underbrace{1\cdots1}_{n_1+1}2\cdots2\,3\cdots3} \\ D_{0\cdots01\cdots1\underbrace{2\cdots2}_{n_2+1}3\cdots3} \\ D_{0\cdots01\cdots1\,2\cdots2\underbrace{3\cdots3}_{n_3+1}} \end{pmatrix} =$$

$$\begin{pmatrix} 2(d+2r+n_1+n_2+n_3-3)\,D_{\underbrace{0\cdots0}_{2(r+1)}1\cdots12\cdots23\cdots3} - C_{0\cdots01\cdots12\cdots23\cdots3}(\hat{D}_0) - 2m_0^2\,D_{0\cdots01\cdots12\cdots23\cdots3} \\ \delta_{n_1 0}\,C_{0\cdots0\underbrace{1\cdots1}_{n_2}\underbrace{2\cdots2}_{n_3}}(\hat{D}_1) - C_{0\cdots01\cdots12\cdots23\cdots3}(\hat{D}_0) - 2n_1\,D_{\underbrace{0\cdots0}_{2(r+1)}\underbrace{1\cdots1}_{n_1-1}2\cdots23\cdots3} - f_1\,D_{0\cdots01\cdots12\cdots23\cdots3} \\ \delta_{n_2 0}\,C_{0\cdots0\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_3}}(\hat{D}_2) - C_{0\cdots01\cdots12\cdots23\cdots3}(\hat{D}_0) - 2n_2\,D_{0\cdots01\cdots1\underbrace{2\cdots2}_{n_2-1}3\cdots3} - f_2\,D_{0\cdots01\cdots12\cdots23\cdots3} \\ \delta_{n_3 0}\,C_{0\cdots0\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}}(\hat{D}_2) - C_{0\cdots01\cdots12\cdots23\cdots3}(\hat{D}_0) - 2n_3\,D_{0\cdots01\cdots12\cdots2\underbrace{3\cdots3}_{n_3-1}} - f_3\,D_{0\cdots01\cdots12\cdots23\cdots3} \end{pmatrix}$$

Discard $k^{th} = \{1, 2, 3\}$ row [not $k=0$].

$$j = \{0, 1, 2, 3 \text{ not } k\}.$$

$$\sum_{m=1}^{3} \left(X_{(k)(0)}\right)_{jm} D_{m0\cdots01\cdots12\cdots23\cdots3}$$

$$= \delta_{j0}\left[2(d+2r+n_1+n_2+n_3-3)\,D_{\underbrace{0\cdots0}_{2(r+1)}1\cdots12\cdots23\cdots3} - C_{0\cdots01\cdots12\cdots23\cdots3}(\hat{D}_0) - 2m_0^2\,D_{0\cdots01\cdots12\cdots23\cdots3}\right]$$

$$+ \bar{\delta}_{j0}\left[\delta_{n_j 0}\,C_{0\cdots0\underbrace{1\cdots1}_{n_{1(j)}}\underbrace{2\cdots2}_{n_{2(j)}}}(\hat{D}_j) - C_{0\cdots01\cdots12\cdots23\cdots3}(\hat{D}_0) - 2n_j\,D_{\hat{j}00\cdots01\cdots12\cdots23\cdots3} - f_j D_{0\cdots01\cdots12\cdots23\cdots3}\right]$$

Invert $\left(X_{(k)(0)}\right)_{jm}$.

identities (4 propagators)

$$\tilde{X}_{00} = 8 \det Z$$

$$\tilde{X}_{0i} = \tilde{X}_{i0} = -\sum_{n=1}^{3} 4 \tilde{Z}_{in} f_n$$

$$\tilde{X}_{ij} = 8 m_0^2 \tilde{Z}_{ij} + \sum_{n,m=1}^{3} 2 \tilde{\tilde{Z}}_{(in)(jm)} f_n f_m$$

$$\tilde{\tilde{X}}_{(0i)(0j)} = -4 \tilde{Z}_{ij}$$

$$\tilde{\tilde{X}}_{(0i)(jk)} = \tilde{\tilde{X}}_{(jk)(0i)} = -2 \sum_{n=1}^{3} f_n \tilde{\tilde{Z}}_{(ni)(jk)} \qquad \text{only for } i,j,k = 1,\dots,3$$

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Useful id:

$$\left(X^{-1}_{(jk)(0)}\right)_{ij} = \frac{\tilde{\tilde{X}}_{(jk)(0i)}}{\tilde{X}_{0k}}$$

$$= \frac{1}{-4 \sum_{n=1}^{3} \tilde{Z}_{in} f_n} \begin{cases} -4 \tilde{Z}_{ik} & \text{if } j=0 \\ -2 \sum_{n=1}^{3} f_n \tilde{\tilde{Z}}_{(ni)(jk)} & \text{if } j = 1,2,3 \end{cases}$$

$$= \frac{1}{\sum_{n=1}^{3} \tilde{Z}_{in} f_n} \left[ \delta_{j0} \tilde{Z}_{ik} + \frac{1}{2}(1-\delta_{j0}) \sum_{n=1}^{3} f_n \tilde{\tilde{Z}}_{(ni)(jk)} \right]$$

In Mathematica, this is called adjXO[[i]]

Multiply by $\left(X^{-1}_{(jk)(0)}\right)_{ij}$ and sum over $j=\{0,1,2,3\}$.

$\rightarrow$ get $\delta_{im}$ in LHS. perform the sum over m.

$$D_{i\,0\cdots01\cdots12\cdots22\cdots3} =$$

$$\left(X^{-1}_{(k)(0)}\right)_{i0}\Big[2(d+2r+n_1+n_2+n_3-3)\, D_{\underbrace{0\cdots0}_{2(r+1)}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}} - C_{\underbrace{0\cdots0}_{2r}1\cdots12\cdots23\cdots3}(\hat D_0) - 2m_0^2 D_{0\cdots01\cdots12\cdots23\cdots3}\Big]$$

$$+\sum_{j=0}^{3}\left(X^{-1}_{(k)(0)}\right)_{ij}\bar\delta_{j0}\Big[\delta_{n_j0}\, C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1(j)}\underbrace{2\cdots2}_{n_2(j)}}(\hat D_j) - C_{0\cdots01\cdots12\cdots23\cdots3}(\hat D_0) - 2n_j D^{\hat j}_{00\,0\cdots01\cdots12\cdots23\cdots3} - f_j D_{0\cdots01\cdots12\cdots23\cdots3}\Big]$$

only from $1\to3$

Insert $\left(X^{-1}_{(k)(0)}\right)_{ij} = \dfrac{\hat{\tilde X}_{(jk)(0i)}}{\tilde X_{0k}}$,

bring $\tilde X_{0k}$ to LHS.

$$\overbrace{\left(\sum_{n=1}^{3}\tilde Z_{in} f_n\right)}^{\text{vanising}} D_{i0\cdots01\cdots12\cdots23\cdots3} =$$

$$\tilde Z_{ik}\Big[2(d+2r+n_1+n_2+n_3-3)\, D_{\underbrace{0\cdots0}_{2(r+1)}1\cdots12\cdots23\cdots3} - C_{0\cdots01\cdots12\cdots23\cdots3}(\hat D_0) - 2m_0^2 D_{0\cdots01\cdots12\cdots23\cdots3}\Big]$$

$$+\frac12\sum_{j=1}^{3}\sum_{n=1}^{3} f_n \tilde{\tilde Z}_{(ni)(jk)}\Big[\delta_{n_j0}\, C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1(j)}\underbrace{2\cdots2}_{n_2(j)}}(\hat D_j) - C_{0\cdots01\cdots12\cdots23\cdots3}(\hat D_0) - 2n_j D^{\hat j}_{000\cdots01\cdots12\cdots23\cdots3}\Big]$$

$$-\frac12\underbrace{\left(\sum_{j=1}^{3}\sum_{n=1}^{3} f_n f_j \tilde{\tilde Z}_{(ni)(jk)}\right)}_{-\frac14\tilde X_{ik} + 2m_0^2\tilde Z_{ik}} D_{0\cdots01\cdots12\cdots23\cdots3}$$

$\leftarrow$ cancels last term of first line.

$\uparrow$ solve.

$$\frac14\tilde X_{ik} D_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_2}}$$

$$= \tilde Z_{ik}\Big[2(d+2r+n_1+n_2+n_3-3)\, D_{\underbrace{0\cdots0}_{2(r+1)}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}} - C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}(\hat D_0)\Big]$$

$$+\frac12\sum_{j=1}^{3}\sum_{n=1}^{3} f_n \tilde{\tilde Z}_{(ni)(jk)}\Big[\delta_{n_j0}\, C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1(j)}\underbrace{2\cdots2}_{n_2(j)}}(\hat D_j) - C_{\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}(\hat D_0) - 2n_j D^{\hat j}_{00\underbrace{0\cdots0}_{2r}\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}\Big]$$

Use this eqn if $r\le 0$.

If $r=0$,

$$D_{\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}} = \frac{4\tilde{Z}_{ik}}{\tilde{X}_{ik}}\Big[\underbrace{\overbrace{2(d+n_1+n_2+n_3-3)}^{2(1+n_1+n_2+n_3)}D_{00\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}}_{\text{not UV div.}} - C_{\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}(\hat{D}_0)\Big]$$

$$+\frac{2}{\tilde{X}_{ik}}\sum_{j=1}^{3}\sum_{n=1}^{3} f_n \tilde{\tilde{Z}}_{(ni)(jk)}\Big[\delta_{n_j,0}\, C_{\underbrace{0\cdots0}_{0}\underbrace{1\cdots1}_{n_{1(j)}}\underbrace{2\cdots2}_{n_{2(j)}}}(\hat{D}_j) - C_{\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}(\hat{D}_0)$$

$$- 2n_j\, D_{\hat{j}00\underbrace{1\cdots1}_{n_1}\underbrace{2\cdots2}_{n_2}\underbrace{3\cdots3}_{n_3}}\Big]$$

[2]

Note: Formula [1] is applicable only if $!(\tilde{Z}_{ij}=0)$ [used for $D_{\underbrace{0\cdots0}_{r>0}1\cdots1\,2\cdots2\,3\cdots3}$]
Formula [2] is applicable only if $!(\tilde{X}_{ij}=0)$ [used for $D_{\underbrace{0\cdots0}_{r\leq0}1\cdots1\,2\cdots2\,3\cdots3}$]

If $\tilde{Z}_{ij}=0$
can obtain reduction formula by direct integration.