Quantum Chromodynamics — Formulation

- Based on group $SU(3)$

Fermions:	Quarks	(fundamental rep.)	$i = 1, 2, 3$
Gauge bosons:	Gluons	(adjoint rep.)	$a = 1, 2, \ldots, 8$

- Interact via exchanging color, the "charge" of QCD.

- Six (known) quark flavors (up, down, charm, strange, top, bottom) with different "masses."

Lagrangian:

$i,j = \{1,2,3\}$ $\qquad$ $a = \{1,2,\ldots 8\}$

$$\mathcal{L}_{QCD} = \sum_f \bar{\psi}_f^{\,i} \left(i \not{D}^{ij} - m_f \right) \psi_f^{\,j} - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} + \underbrace{\frac{g^2}{64\pi^2} \theta_{QCD} \epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma}}_{\frac{g^2}{32\pi^2} \theta_{QCD} G^a_{\mu\nu} \tilde{G}^{a\mu\nu}}$$

$\psi_f \equiv$ quark fields — in fundamental representation.

$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu \equiv$ field strength tensor.

(can be written $G^a_{\mu\nu} T^a = \frac{i}{g} [D_\mu, D_\nu]$)

$D_\mu \psi = \left(\partial_\mu + i g T^a A^a_\mu \right) \psi \equiv$ covariant derivative.
(in the fundamental rep)

$A^a_\mu \equiv$ gauge fields

$g \equiv$ strong coupling constant — sometimes written g_3
↳ define $\alpha_s = g^2/4\pi$

often convenient to extract another factor of π:

$$a = \frac{\alpha_s}{\pi}.$$

Like any typical quantum field theory, all of these quantities will have to be renormalized in some particular scheme (usually $\overline{MS}$).

The generators of SU(3), T^a, satisfy the SU(3) Lie algebra:

$$[T^a, T^b] = i f^{abc} T^c$$

structure constants. (totally antisymmetric)

In the standard representation, the generators, T^a, are repesented by Gell-Mann matrices, λ^a: $T^a = \frac{1}{2}\lambda^a$.

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & \\ 0 & -1 & \\ & & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(SINGLET)

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \qquad \lambda_0 = \sqrt{\tfrac{2}{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

All hermitian, and traceless. Two are diagonal, λ_3 & λ_8, and commute with each other $\Rightarrow$ rank of SU(3) is 2.

Structure constants in this representation: $f^{abc} = -2i\,\mathrm{Tr}\left([T^a, T^b]\,T^c\right)$

$$f_{123} = 1, \qquad f_{458} = f_{678} = \sqrt{3}/2$$

$$f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2}.$$

The structure constants, themselves, provide a representation of the algebra, if we take $(T^a)^{bc} \equiv -i f^{abc}$ (proven in P. Stevenson's HW 6 #2).

– used the Jacobi Identity.

Covariant Derivative:

Always $(D_\mu)_{ij} = \delta_{ij}\,\partial_\mu + ig\,T^a_{ij} A^a_\mu$ ← any representation T^a_{ij}

acting on an adjoint index: change $ij \to a,b$ rename $a \to c$

$$(D_\mu)^{ab} = \delta^{ab}\,\partial_\mu + ig\,(T^c)^{ab} A^c_\mu \qquad \text{but} \quad (T^c)^{ab} = -i f^{cab}$$

$$= \delta^{ab}\,\partial_\mu + g\, f^{cab} A^c_\mu$$

$$= \delta^{ab}\,\partial_\mu + g f^{abc} A^c_\mu$$

Commutator:

$$[D_\mu, D_\nu] = [\partial_\mu + igT^a A^a_\mu,\ \partial_\nu + igT^b A^b_\nu]$$

$$= \underbrace{igT^c\,(\partial_\mu A^c_\nu - \partial_\nu A^c_\mu)}_{\text{(from abelian case)}} - g^2 A^a_\mu A^b_\nu [T^a, T^b]$$

$$= igT^c\,(\partial_\mu A^c_\nu - \partial_\nu A^c_\mu) - g^2 A^a_\mu A^b_\nu\, i f^{abc}\, T^c$$

$$= igT^c \underbrace{\left[(\partial_\mu A^c_\nu - \partial_\nu A^c_\mu) - g f^{abc} A^b_\mu A^c_\nu\right]}_{F^c_{\mu\nu}}$$

$$= igT^c F^c_{\mu\nu}$$

[When dealing with matrix-valued fields, one defines $igT^c F^c_{\mu\nu} = \underbrace{g f^{abc} F^c_{\mu\nu}}_{\text{(in adj. rep)}} = (F_{\mu\nu})^{ab}$]