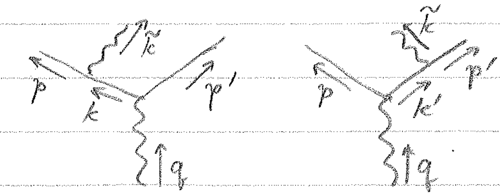
Kinematics and the Origin of the IR singularity:

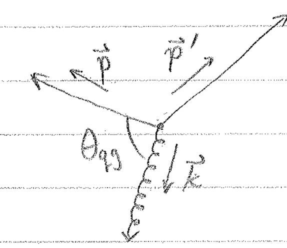
Look at denominator of quark propagators (include the quark masses for now)



$$\frac{i(\tilde{\not{k}}+\not{p}+m)}{(\tilde{k}+p)^2-m^2} \quad \& \quad \frac{i(\tilde{\not{k}}+\not{p}'+m)}{(\tilde{k}+p')^2-m^2}$$

$$\text{Denom} = (\tilde{k}+p)^2 - m^2 = \underset{0}{\underset{\|}{\tilde{k}^2}} + \underset{\cancel{m^2}}{\underset{\|}{p^2}} + 2\tilde{k}\cdot p - \cancel{m^2} = 2\tilde{k}\cdot p$$

Let $\tilde{k}^\mu = (E_g, \vec{k})$ 4-momentum of gluon
$p^\mu = (E_q, \vec{p})$ 4-momentum of quark



$$\therefore \text{Denom} = 2(E_g E_q - \vec{k}\cdot\vec{p})$$
$$= 2(E_g E_q - |\vec{k}||\vec{p}|\cos\theta_{qg})$$

Use: $|\vec{k}| = E_g$ (massless gluon)
$|\vec{p}| = \sqrt{E_q^2 - m^2} = E_q\sqrt{1 - m^2/E_q^2}$

$$\text{Denom} = 2\left(E_g E_q - E_g E_q\sqrt{1-\frac{m^2}{E_q^2}}\cos\theta_{qg}\right)$$
$$= 2E_g E_q\left(1 - \sqrt{1-\frac{m^2}{E_q^2}}\cos\theta_{qg}\right)$$

Two sources of IR divergences:

① $E_g \to 0$ <u>soft gluon singularity</u>

② If $m=0$ (massless quarks), $\sqrt{1-\frac{m^2}{E_q^2}} = 1$

$\theta_{qg} \to 0$ <u>collinear</u> or <u>mass singularity</u>

Infrared Safety

In QCD, would like to choose μ_R as "large as possible" in perturbative calculations. $\longrightarrow$ small $g(\mu_R)$.

$\Rightarrow$ Then, we should look at observables at high energy

Difficulty is that there are other scales:

Typical S-matrix element

$$S = S\Big(\underbrace{\frac{Q_i^2}{\mu^2}}_{\substack{\text{Large external}\\ \text{momenta}}}, \underbrace{\frac{p_i^2}{\mu^2}}_{\substack{\text{small}\\ \text{external}\\ \text{momenta}}}, \underbrace{\frac{m_f^2}{\mu^2}, \frac{m_g^2}{\mu^2}}_{\substack{\text{'light'}\\ \text{quark/gluon}\\ \text{masses}}}, \frac{Q_i^2}{Q_j^2}, g(\mu)\Big) \quad \ldots$$

[parametric dep. on gluon mass for illustration].

$$= \sum_n g^{2n}(\mu)\; a_n\left(\frac{Q_i^2}{\mu^2}, \ldots\right)$$

These expansion coefficients depend logarithmically on all ratios.
(this is standard lore: see Sterman's PRD vol 17, 2773 (1978))

If we set $\mu^2 = Q_i^2$, then we have $\ln\left(\frac{p_i^2}{\mu^2}\right)$, $\ln\left(\frac{m^2}{\mu^2}\right)$, ...

Infrared safe observables are ones which are independent of these small parameters p_i^2, m_f^2, m_g^2, ...