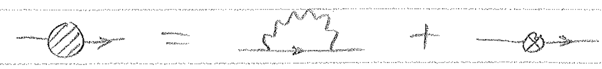
Calculating the residue of pole, Z_{res}: given by: $\left(1 - \frac{d\Sigma}{d\not{p}}\Big|_{m_{phys}^2}\right)^{-1}$

$-i\Sigma(\not{p}) =$ 

In Feynman gauge, $\xi = 1$, this is $(\overline{ms})$

$$= \frac{ig^2 C_F}{(4\pi)^2}\left[-\not{p}\left(1 + \int_0^1 dx\ 2(1-x)\ln\left(\frac{-x(1-x)p^2 + xm^2}{\mu^2}\right)\right)\right.$$
$$\left. + m\left(2 + 4\int_0^1 dx\ \ln\left(\frac{-x(1-x)p^2 + xm^2}{\mu^2}\right)\right)\right]$$

Notice: For massless quarks, $m=0$, taking derivatives and setting $\not{p} = m = 0$ causes the logs to diverge $\ln(0) = $ undef. — this is due to an IR singularity.
— Back up: use expression for general d:

$$-i\Sigma(\not{p}) = \frac{ig^2 C_F}{(4\pi)^2}\int_0^1 dx\left[(d-2)(1-x)\not{p}\right]\mu^{2\epsilon}\Gamma\left(2-\frac{d}{2}\right)\left(\frac{1}{-x(1-x)p^2}\right)^{2-\frac{d}{2}} + i(\not{p}\delta_\psi - \delta_m)$$

To obtain the counterterm, we needed the UV singularity. We took $d = 4-2\epsilon$, with ϵ positive (less than 4 dim.) and found the $\frac{1}{\epsilon}$ pole. Having fixed the counterterm, the diagram is free of UV divergences — we may now take $\epsilon < 0$ (more than 4 dim) to extract the IR singularity when $m=0$, and when $\not{p}$ is set to $m=0$.

$$-i\frac{d\Sigma(\not{p})}{d\not{p}} = \frac{ig^2 C_F}{(4\pi)^2}\int_0^1 dx\left\{\left[(d-2)(1-x)\right]\mu^{2\epsilon}\Gamma\left(2-\frac{d}{2}\right)\left(\frac{1}{-x(1-x)p^2}\right)^{2-\frac{d}{2}}\right.$$
$$\left. + \left[(d-2)(1-x)\not{p}\right]\mu^{2\epsilon}\Gamma\left(2-\frac{d}{2}\right)\not{p}\,\frac{(d-4)}{2}\left(\frac{1}{-x(1-x)}\right)^{2-\frac{d}{2}}\left(\frac{1}{p^2}\right)^{3-\frac{d}{2}}\right\}$$

combine

$$= \frac{ig^2 C_F}{(4\pi)^2}\int_0^1 dx\left(1 + \frac{d-4}{2}\right)\left[(d-2)(1-x)\right]\mu^{2\epsilon}\Gamma\left(2-\frac{d}{2}\right)\left(\frac{1}{-x(1-x)p^2}\right)^{2-\frac{d}{2}} + i\delta_\psi$$

Now, setting $p^2 = m^2 = 0$, since the exponent $2 - \frac{d}{2} = +\epsilon < 0$, the $-x(1-x)p^2$ is effectively in the numerator, and the integral vanishes.

We're left with

$$-i\frac{d\Sigma(\not p)}{d\not p}\bigg|_{\not p=m=0} = i\,\delta_\psi \qquad \Rightarrow \qquad \frac{d\Sigma(\not p)}{d\not p}\bigg|_{\not p=m=0} = -\delta_\psi$$

$$= \frac{i g^2}{(4\pi)^2}(-C_F)\left(\frac{1}{\epsilon}-\gamma_E+\ln 4\pi\right) \leftarrow \text{this is the IR divergent part.}$$

What's going on? In the massless, on-shell limit, the self-energy diagram vanishes since the dimensionally regulated integral has no scale — the UV divergent and IR divergent parts of the same diagram exactly cancel.

$$\text{Essentially:}\quad -i\Sigma'(\not p) = \underbrace{\begin{pmatrix}\text{U.V.}\\ \text{div}\end{pmatrix} + \begin{pmatrix}\text{I.R.}\\ \text{div}\end{pmatrix}}_{\text{cancel}} + \begin{pmatrix}\delta_{C.T}\\ \text{U.V. div}\end{pmatrix}$$

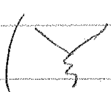
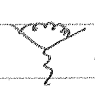
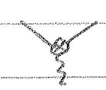
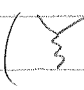
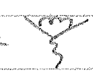
designed to cancel $\Rightarrow \begin{pmatrix}\text{IR}\\ \text{div}\end{pmatrix} = \begin{pmatrix}\delta_{C.T.}\\ \text{U.V. div}\end{pmatrix}$

Hence, the residue of the quark pole is IR divergent:

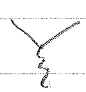
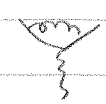
$$Z_{res} = \left(1 + \frac{g^2}{(4\pi)^2}(-C_F)\left(\frac{1}{\epsilon}-\gamma_E+\ln 4\pi\right)\right)^{-1} = (1+\delta_\psi)^{-1}$$

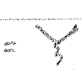
$$\approx 1 - \frac{g^2}{(4\pi)^2}(-C_F)\left(\frac{1}{\epsilon}-\gamma_E+\ln 4\pi\right) = 1-\delta_\psi$$

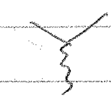
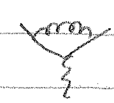
So, now the matrix element for $e^+e^- \to q\bar q$ at NLO (virtual correction) is:

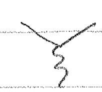
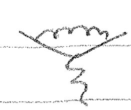
$Z_{res} \times \Big($  $+$  $+$  $\Big) = (1-\delta_\psi)\Big($  $+$  $+$  $\Big)$

from LSZ

$=$  $(1-\delta_\psi) +$  $+$ 

But  $=$  $\delta_{\gamma\bar\psi\psi}$

$=$  $\big(1 - \underbrace{\delta_\psi + \delta_{\gamma\bar\psi\psi}}_{\text{cancel since } Z_\psi = Z_{\gamma\bar\psi\psi} \text{ in QED}}\big) +$ 

$=$  $+$ . All UV div. gone $\Rightarrow$ left with IR div.