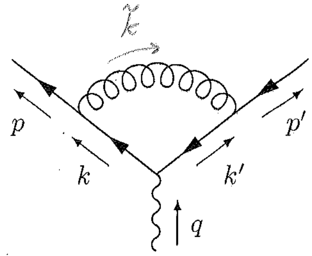
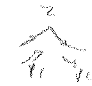
Evaluation of the Quark Vertex Function — Virtual Correction



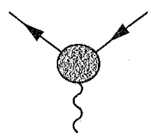


Kinematic identities:

$$q^\mu = (p+p')^\mu \quad \Rightarrow \quad q^2 = p^2 + p'^2 + 2p\cdot p' = 2p\cdot p'$$
$$k'^\mu = (p' - \tilde{k})^\mu$$
$$k^\mu = (p+\tilde{k})^\mu$$

This is a 2nd order QCD correction to the QED vertex:

$ig\mu^\epsilon T^a_{ik}\Gamma^\mu$ for photon → gluon.



$$\equiv \bar{u}_i^{(s)}(p)(iQe\mu^\epsilon\delta_{ik}\Gamma^\mu)v_k^{(s')}(p')\,. \tag{2}$$

Use the Feynman rules to write down the matrix element in $d = 4-2\epsilon$ dimensions. (The dim. reg. ϵ and Feynman prescription $i\epsilon$ inside propagators are not to be confused).

$$i\mathcal{M} = \bar{u}_i^{(s)}(p)(ig\mu^\epsilon\gamma^\nu T^a_{ij})\int\frac{d^d\tilde{k}}{(2\pi)^d}\left\{\frac{i\delta_{jk}\not{k}}{k^2+i\epsilon}(iQe\mu^\epsilon\gamma^\mu)\frac{i\delta_{k\ell}(-\not{k}')}{k'^2+i\epsilon}\right.$$
$$\left.\times\left[\frac{-i\delta^{ab}(g_{\nu\rho}-(1-\xi)\tilde{k}_\nu\tilde{k}_\rho/\tilde{k}^2)}{\tilde{k}^2+i\epsilon}\right]\right\}(ig\mu^\epsilon\gamma^\rho T^b_{\ell m})v_m^{(s')}(p')$$
$$= \bar{u}_i^{(s)}(p)(iQe\mu^\epsilon)\underbrace{T^a_{ij}T^a_{jk}}_{C_F\delta_{ik}}\left[ig^2\mu^{2\epsilon}\int\frac{d^d\tilde{k}}{(2\pi)^d}\frac{\gamma^\nu\not{k}\gamma^\mu\not{k}'\gamma^\rho(g_{\nu\rho}-(1-\xi)\tilde{k}_\nu\tilde{k}_\rho/\tilde{k}^2)}{(k^2+i\epsilon)(k'^2+i\epsilon)(\tilde{k}^2+i\epsilon)}\right]v_k^{(s')}(p')$$

Comparing with (eqn 2), the factor inside the square braces is the 2nd order vertex form factor, $\Gamma^\mu_{[2]}$, which needs to be evaluated.

$$\delta\Gamma^\mu_{[2]} = ig^2C_F\mu^{2\epsilon}\int\frac{d^d\tilde{k}}{(2\pi)^d}\left\{\frac{\gamma^\nu\not{k}\gamma^\mu\not{k}'\gamma^\rho(g_{\nu\rho}-(1-\xi)\tilde{k}_\nu\tilde{k}_\rho/\tilde{k}^2)}{(k^2+i\epsilon)(k'^2+i\epsilon)(\tilde{k}^2+i\epsilon)}\right\}$$
$$= ig^2C_F\mu^{2\epsilon}\int\frac{d^d\tilde{k}}{(2\pi)^d}\left\{\underbrace{\frac{\gamma^\nu\not{k}\gamma^\mu\not{k}'\gamma^\rho g_{\nu\rho}}{(k^2+i\epsilon)(k'^2+i\epsilon)(\tilde{k}^2+i\epsilon)}}_{\text{(gauge indep. term)}} - (1-\xi)\underbrace{\frac{\gamma^\nu\not{k}\gamma^\mu\not{k}'\gamma^\rho\tilde{k}_\nu\tilde{k}_\rho}{(k^2+i\epsilon)(k'^2+i\epsilon)\tilde{k}^2(\tilde{k}^2+i\epsilon)}}_{\text{(gauge dep. term)}}\right\} \tag{3}$$

The real part of $\delta F_1(q^2)$ comes out gauge-independent.

Combine denominators following Feynman's trick:

$$\frac{1}{(k^2+i\epsilon)(k'^2+i\epsilon)(\tilde{k}^2+i\epsilon)} = \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{2}{(\ldots\ldots)^3}$$
$$\frac{1}{(k^2+i\epsilon)(k'^2+i\epsilon)\tilde{k}^2(\tilde{k}^2+i\epsilon)} = \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{6z}{(\ldots\ldots)^4}$$

where
$$(\ldots\ldots) = xk^2 + yk'^2 + z\tilde{k}^2 + (x+y+z)i\epsilon$$
$$= x(p+\tilde{k})^2 + y(p'-\tilde{k})^2 + z\tilde{k}^2 + i\epsilon$$
$$= \cancel{xp^2}^{\,m^2=0} + 2xp\cdot\tilde{k} + x\tilde{k}^2 + \cancel{yp'^2}^{\,m^2=0} - 2yp\cdot\tilde{k} + y\tilde{k}^2 + z\tilde{k}^2 + i\epsilon$$
$$= (x+y+z)\tilde{k}^2 + 2xp\cdot\tilde{k} - 2yp'\cdot\tilde{k}$$

complete the square: $\ell = \tilde{k} + xp - yp' \;\Rightarrow\; \underline{\tilde{k} = \ell - xp + yp'}$

$\gamma_\mu \gamma^\mu = d$

$\gamma_\mu \gamma^\rho \gamma^\mu = \underset{4-2\epsilon}{(2)}\gamma^\rho$

$$\Rightarrow\; \tilde{k}^2 + 2(xp - yp')\tilde{k} + (xp - yp')^2$$

$$\begin{aligned}
(\ldots\ldots) &= \ell^2 - (xp - yp')^2 \\
&= \ell^2 - x^2p^2 - \overset{m^2=0}{\cancel{y^2p'^2}} + 2xyp\cdot p' \\
&= \ell^2 + xyq^2 \\
&= \ell^2 - \Delta \qquad \text{with } \underline{\Delta = -xyq^2}, \text{ for both terms.}
\end{aligned}$$

This gives us

$$\delta\Gamma^\mu_{[2]} = ig^2C_F\mu^{2\epsilon}\overset{\int_F}{\int}\frac{d^d\tilde{k}}{(2\pi)^d}\left\{\underbrace{\frac{2\,(\gamma^\nu \not{k}\gamma^\mu \not{k}'\gamma^\rho g_{\nu\rho})}{(\ell^2-\Delta)^3}}_{\text{(gauge indep. term)}} - (1-\xi)\underbrace{\frac{6z\,(\gamma^\nu \not{k}\gamma^\mu \not{k}'\gamma^\rho \tilde{k}_\nu \tilde{k}_\rho)}{(\ell^2-\Delta)^4}}_{\text{(gauge dep. term)}}\right\}. \tag{4}$$

Simplify <u>gauge independent</u> numerator (Dirac algebra):

$$\begin{aligned}
2\,(\gamma^\nu \not{k}\gamma^\mu \not{k}'\gamma^\rho g_{\nu\rho}) &= 2\,\gamma^\nu \not{k}\gamma^\mu \not{k}'\gamma_\nu = 2\,k_\sigma k'_\lambda \gamma^\nu\gamma^\sigma\gamma^\mu\gamma^\lambda\gamma_\nu \\
&= 2\,k_\sigma k'_\lambda(-2\gamma^\lambda\gamma^\mu\gamma^\sigma + 2\epsilon\gamma^\sigma\gamma^\mu\gamma^\lambda) \\
&= -4\not{k}'\gamma^\mu\not{k} + 4\epsilon\,\not{k}\gamma^\mu\not{k}' \\
&= -4(\not{p}' - \tilde{\not{k}})\gamma^\mu(\not{p} + \tilde{\not{k}}) + 4\epsilon\,(\not{p} + \tilde{\not{k}})\gamma^\mu(\not{p}' - \tilde{\not{k}}) \\
&= -4(\not{p}'\gamma^\mu\not{p} + \not{p}'\gamma^\mu\tilde{\not{k}} - \tilde{\not{k}}\gamma^\mu\not{p} - \tilde{\not{k}}\gamma^\mu\tilde{\not{k}}) + 4\epsilon\,(\overset{\bar{u}(p)\not{p}=0}{\cancel{\not{p}\gamma^\mu\not{p}'} - \cancel{\not{p}\gamma^\mu\tilde{\not{k}}}} + \overset{\not{p}'v(p)=0}{\cancel{\tilde{\not{k}}\gamma^\mu\not{p}'}} - \tilde{\not{k}}\gamma^\mu\tilde{\not{k}}) \\
&= -4(\underset{\mathbf{I}}{\not{p}'\gamma^\mu\not{p}} + \underset{\mathbf{II}}{\not{p}'\gamma^\mu\tilde{\not{k}}} - \underset{\mathbf{III}}{\tilde{\not{k}}\gamma^\mu\not{p}} - \underset{\mathbf{IV}}{\tilde{\not{k}}\gamma^\mu\tilde{\not{k}}}) - 4\epsilon\,\tilde{\not{k}}\gamma^\mu\tilde{\not{k}}
\end{aligned}$$

Simplify all four terms (order ϵ term is identical to **IV**):

1. Push $\not{p}$ to the left and $\not{p}'$ to the right using the anticommutation relation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.
2. Use the Dirac equation $\bar{u}(p)\not{p} = \not{p}'v(p') = 0$ to reduce expression.
3. By symmetric integration, term with odd powers of vector ℓ^μ vanish. Similarly, an integration over the bilinear $\ell^\mu\ell^\nu$ reduces to $\ell^\mu\ell^\nu \to \dfrac{1}{d}\ell^2g^{\mu\nu}$.

$$\begin{aligned}
(\mathbf{I}):\quad \not{p}'\gamma^\mu\not{p} &= 2p'^\mu\not{p} - \gamma^\mu\not{p}'\not{p} \\
&= \overset{0}{\cancel{2p'^\mu\not{p}}} - 2\gamma^\mu p'\cdot p + \overset{0}{\cancel{\gamma^\mu\not{p}\not{p}'}} \\
&= -\gamma^\mu q^2 \;\; \text{(kin. iden. 1)}
\end{aligned}$$

$$\begin{aligned}
(\mathbf{II}):\quad \not{p}'\gamma^\mu\tilde{\not{k}} &= 2p'^\mu\tilde{\not{k}} - \gamma^\mu\not{p}'\tilde{\not{k}} \\
&= 2p'^\mu\tilde{\not{k}} - 2\gamma^\mu p'\cdot\tilde{k} + \overset{0}{\cancel{\gamma^\mu\tilde{\not{k}}\not{p}'}} \\
&= 2p'^\mu(\overset{\text{odd}}{\cancel{\not{\ell}}} - \overset{0}{\cancel{x\not{p}}} + \overset{0}{\cancel{y\not{p}'}})p^\mu - 2\gamma^\mu p'\cdot\not{k} \\
&= -2\gamma^\mu p'\cdot(\ell - xp + \overset{p'^2=0}{\cancel{yp'}}) \\
&= 2x\gamma^\mu p'\cdot p = x\gamma^\mu q^2
\end{aligned}$$

$$\begin{aligned}
(\mathbf{III}):\quad -\tilde{\not{k}}\gamma^\mu\not{p} &= -2\tilde{\not{k}}p^\mu + \tilde{\not{k}}\not{p}\gamma^\mu \\
&= -2\tilde{\not{k}}p^\mu + 2\tilde{k}\cdot p\gamma^\mu - \overset{0}{\cancel{\not{p}\tilde{\not{k}}\gamma^\mu}} \\
&= -2(\overset{\text{odd}}{\cancel{\not{\ell}}} - \overset{0}{\cancel{x\not{p}}} + \overset{0}{\cancel{y\not{p}'}})p^\mu \\
&\quad + 2(\overset{\text{odd}}{\cancel{\ell}} - \overset{p^2=0}{\cancel{xp}} + yp') \\
&= 2y\gamma^\mu p\cdot p' \\
&= y\gamma^\mu q^2
\end{aligned}$$

$$
\begin{aligned}
(\mathbf{IV}), \mathcal{O}(\epsilon): \quad -\tilde{\not{k}}\gamma^\mu\tilde{\not{k}} &= -2\tilde{k}^\mu\tilde{\not{k}} + \gamma^\mu\tilde{k}^2 \\
&= -2(\ell - xp + yp')^\mu(\not{\ell} - \overset{0}{\cancel{x\not{p}}} + \overset{0}{\cancel{y\not{p}'}}) + \gamma^\mu(\ell - xp + yp')^2 \\
&= -2\ell^\mu\not{\ell} + \cancel{(\text{odd } \ell)} + \gamma^\mu(\ell^2 + (xp - yp')^2 + \cancel{(\text{odd } \ell)}) \\
&= -2\ell^\mu\ell^\nu\gamma_\nu + \gamma^\mu\ell^2 + \gamma^\mu(-2xyp\cdot p') + \gamma^\mu(\cancel{x^2p^2} + \cancel{y^2p'^2}) \\
\text{by symmetric integration} \quad &\to -2\frac{1}{d}\ell^2 g^{\mu\nu}\gamma_\nu + \gamma^\mu\ell^2 - xy\gamma^\mu q^2 \\
&= \left(\tfrac{-2}{d} + 1\right)\gamma^\mu\ell^2 - xy\gamma^\mu q^2
\end{aligned}
$$

The gauge independent numerator becomes

$$
\begin{aligned}
2\,(\gamma^\nu\not{k}\gamma^\mu\not{k}'\gamma^\rho g_{\nu\rho}) &= -4\big(-\gamma^\mu q^2 + x\gamma^\mu q^2 + y\gamma^\mu q^2 + (\tfrac{-2}{d}+1)\gamma^\mu\ell^2 - xy\gamma^\mu q^2)\big) \\
&\qquad + 4\epsilon\,\left[(\tfrac{-2}{d}+1)\gamma^\mu\ell^2 - xy\gamma^\mu q^2\right] \\
&= 2(\tfrac{4}{d} - 2)\gamma^\mu\ell^2 + (4z + 4xy)\gamma^\mu q^2 + 4\epsilon\,\left[(\tfrac{-2}{d}+1)\gamma^\mu\ell^2 - xy\gamma^\mu q^2\right] \\
&= 2\left(\tfrac{4}{d} - 2 - \tfrac{4\epsilon}{d} + 2\epsilon\right)\gamma^\mu\ell^2 + (4z + 4xy - 8xy\epsilon)\gamma^\mu q^2 \\
&= -\frac{8(1-\epsilon)^2}{d}\gamma^\mu\ell^2 + 4(z + (1-2\epsilon)xy)\gamma^\mu q^2\,.
\end{aligned}
$$

Hence, the vertex function takes form (gauge dependent term included for completeness)

$$
\begin{aligned}
\delta\Gamma^\mu_{(2)}(q^2) &= ig^2C_F\mu^{2\epsilon}\gamma^\mu\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \\
&\times\int\frac{d^d\ell}{(2\pi)^d}\Bigg\{\frac{\frac{-8}{d}(1-\epsilon)^2\ell^2 + 4(z+(1-2\epsilon)xy)q^2}{(\ell^2-\Delta)^3} \\
&\qquad - (1-\xi)\frac{6z\ell^4 + 3(3z - 6xy + 2)zq^2\ell^2 + 6(-xy + x^2y + xy^2)z(q^2)^2}{(\ell^2-\Delta)^4}\Bigg\}. \qquad (5)
\end{aligned}
$$

Drop gauge dependent term by choosing the Feynman gauge, $\eta = 0$. Decompose the vertex function into its form factors, $\Gamma^\mu = \gamma^\mu F_1(q^2) + \frac{1}{2m}\sigma^{\mu\nu}q_\nu F_2(q^2)$. By inspection[1], $F_2(q^2) = 0$.

$$
\begin{aligned}
\delta F_1(q^2) &= ig^2C_F\mu^{2\epsilon}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int\frac{d^d\ell}{(2\pi)^d}\left\{\frac{\frac{-8}{d}(1-\epsilon)^2\ell^2 + 4(z+(1-2\epsilon)xy)q^2}{(\ell^2-\Delta)^3}\right\} \\
&= ig^2C_F\mu^{2\epsilon}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\Bigg\{\underbrace{\int\frac{d^d\ell}{(2\pi)^d}\frac{\frac{-8}{d}(1-\epsilon)^2\ell^2}{(\ell^2-\Delta)^3}}_{\text{UV divergent}} + \underbrace{\int\frac{d^d\ell}{(2\pi)^d}\frac{4(z+(1-2\epsilon)xy)q^2}{(\ell^2-\Delta)^3}}_{\text{UV convergent}}\Bigg\}
\end{aligned}
\qquad (6)
$$

Integrate over the unconstrained momentum ℓ^μ in the first integral,

$$
\begin{aligned}
\int\frac{d^d\ell}{(2\pi)^d}\frac{\frac{-8}{d}(1-\epsilon)^2\ell^2}{(\ell^2-\Delta)^3} &= -\frac{8}{d}(1-\epsilon)^2\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{(\ell^2-\Delta)^3} \\
&= -\frac{\overset{2}{\cancel{8}}}{\cancel{d}}(1-\epsilon)^2\left(\frac{(-1)^{3-1}i}{(4\pi)^{d/2}}\frac{\cancel{d}}{\cancel{2}}\frac{\Gamma(3-\frac{d}{2}-1)}{\cancel{\Gamma(3)}}\left(\frac{1}{\Delta}\right)^{3-\frac{d}{2}-1}\right) \\
&= -2i(1-\epsilon)^2\frac{1}{(4\pi)^{2-\epsilon}}\Gamma(\epsilon)\left(\frac{1}{-xyq^2}\right)^\epsilon,
\end{aligned}
$$

UV divergence.

$= -\frac{i}{8\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi\right)$

$\delta_{A\psi\psi} = \frac{g^2}{16\pi^2}C_F\left(\frac{1}{\epsilon} - \gamma_E + \ln\right.$

This UV divergence is to be interpreted as an IR divergence from and

[1]That $F_2(q^2)$ vanishes is a consequence of massless fermions.

massless fermions have no magnetic moment.

and in the second integral,

$$\begin{aligned}
\int \frac{d^d\ell}{(2\pi)^d}\frac{4(z+(1-2\epsilon)xy)q^2}{(\ell^2-\Delta)^3} &= 4(z+(1-2\epsilon)xy)q^2\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^3}\\
&= 4(z+(1-2\epsilon)xy)q^2\left(\frac{(-1)^3 i}{(4\pi)^{d/2}}\frac{\Gamma(3-\frac{d}{2})}{\Gamma(3)}\left(\frac{1}{\Delta}\right)^{\overset{3}{n}-\frac{d}{2}}\right)\\
&= \frac{-\overset{2}{\cancel{4}}i}{(4\pi)^{2-\epsilon}}(z+(1-2\epsilon)xy)q^2\frac{\Gamma(1+\epsilon)}{\cancel{2}}\left(\frac{1}{-xyq^2}\right)^{1+\epsilon},
\end{aligned}$$

to arrive at

$$\begin{aligned}
\delta F_1(q^2) = ig^2\mu^{2\epsilon}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\Big\{&-2i(1-\epsilon)^2\frac{1}{(4\pi)^{2-\epsilon}}\Gamma(\epsilon)\left(\frac{1}{-xyq^2}\right)^\epsilon\\
&+\frac{-2i}{(4\pi)^{2-\epsilon}}(z+(1-2\epsilon)xy)q^2\Gamma(1+\epsilon)\left(\frac{1}{-xyq^2}\right)^{1+\epsilon}\Big\}.
\end{aligned} \tag{7}$$

Factor $-i/(4\pi)^{2-\epsilon}$ out of the integral; integrate over z, fixing $z \to 1-x-y$ in second term.

$$\begin{aligned}
&= \frac{g^2\mu^{2\epsilon}}{(4\pi)^{2-\epsilon}}\int_0^1 dx\int_0^{1-x}dy\Big\{2(1-\epsilon)^2\Gamma(\epsilon)\left(\frac{1}{-xyq^2}\right)^\epsilon\\
&\qquad + 2(1-x-y+(1-2\epsilon)xy)q^2\,\Gamma(1+\epsilon)\left(\frac{1}{-xyq^2}\right)^{1+\epsilon}\Big\}\\
&= \frac{g^2\mu^{2\epsilon}}{(4\pi)^{2-\epsilon}}\int_0^1 dx\int_0^{1-x}dy\Big\{2(2-\epsilon)^2\Gamma(\epsilon)\left(\frac{1}{-xyq^2}\right)^\epsilon\\
&\qquad + 2((1-x)(1-y)-2\epsilon\,xy)q^2\,\Gamma(1+\epsilon)\left(\frac{1}{-xyq^2}\right)^{1+\epsilon}\Big\}
\end{aligned} \tag{8}$$

Integrating over Feynman parameters must be done carefully. Since q^μ is a timelike vector, $q^2 > 0$. Pull $(-1)^{-\epsilon}$ factors out of fractions of the form $\left(\frac{1}{-q^2}\right)^\epsilon$. The strategy here is to factorize the parameters and get them in the form of Euler beta functions: $\dfrac{\Gamma(1+x)\Gamma(1+y)}{\Gamma(2+x+y)} = \displaystyle\int_0^1 t^x(1-t)^y\,dt$. Consider the first integral:

$$\int_0^1 dx\int_0^{1-x}dy\,2(1-\epsilon)^2\Gamma(\epsilon)\left(\frac{1}{-xyq^2}\right)^\epsilon = 2(1-\epsilon)^2\Gamma(\epsilon)(-1)^{-\epsilon}\int_0^1 dx\int_0^{1-x}dy\left(\frac{1}{xyq^2}\right)^\epsilon$$

$$\begin{array}{rccc}
\text{Change variables} & y=(1-x)\tilde{y} & \Rightarrow & dy=(1-x)d\tilde{y}\\
\text{Range of integration} & y:[0,\,1-x] & \longrightarrow & \tilde{y}:[0,\,1]
\end{array}$$

$$\begin{aligned}
&\longrightarrow\quad 2(1-\epsilon)^2\,\Gamma(\epsilon)(-1)^{-\epsilon}\int_0^1 dx\int_0^1 d\tilde{y}(1-x)\left(\frac{1}{x(1-x)\tilde{y}q^2}\right)^\epsilon\\
&=\quad 2(1-\epsilon)^2\,\Gamma(\epsilon)(q^2)^{-\epsilon}(-1)^{-\epsilon}\int_0^1 dx\,x^{-\epsilon}(1-x)^{1-\epsilon}\int_0^1 d\tilde{y}\,\tilde{y}^{-\epsilon}(1-\tilde{y})^0\\
&=\quad 2(1-\epsilon)^2\,\Gamma(\epsilon)(q^2)^{-\epsilon}(-1)^{-\epsilon}\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}\frac{\Gamma(1-\epsilon)\Gamma(1)}{\Gamma(2-\epsilon)}\\
&=\quad 2(1-\epsilon)^2\,\Gamma(\epsilon)(q^2)^{-\epsilon}(-1)^{-\epsilon}\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}\frac{\cancel{\Gamma(1-\epsilon)}}{(1-\epsilon)\cancel{\Gamma(1-\epsilon)}} \qquad \text{(iden: } \Gamma(z+1)=z\Gamma(z))
\end{aligned}$$

$\Gamma(1-\frac{d}{2}) = -\frac{d}{2}\Gamma(-\frac{d}{2})$

The second integral follows similarly:

$$\int_0^1 dx \int_0^{1-x} dy\, 2\big((1-x)(1-y) - 2\epsilon\, xy\big) q^2\, \Gamma(1+\epsilon)\left(\frac{1}{-xyq^2}\right)^{1+\epsilon} =$$

$$2Q^2\Gamma(1+\epsilon)(-1)^{1-\epsilon}\int_0^1 dx \int_0^{1-x} dy\big((1-x)(1-y) - 2\epsilon\, xy\big)\left(\frac{1}{xyq^2}\right)^{1+\epsilon}$$

Under the same change of variables introduced for the first integral, the second integral decouples, albeit into three terms:

$$\longrightarrow -2q^2\Gamma(1+\epsilon)(-1)^{-\epsilon}\int_0^1 dx\int_0^1 d\tilde{y}(1-x)\Big((1-x)(1-(1-x)\tilde{y}) - 2\epsilon\, x(1-x)\tilde{y}\Big)\left(\frac{1}{x(1-x)\tilde{y}q^2}\right)^{1+\epsilon}$$

$$= -2(q^2)^{-\epsilon}\Gamma(1+\epsilon)(-1)^{-\epsilon}\Bigg(\int_0^1 dx\, x^{-1-\epsilon}(1-x)^{1-\epsilon}\int_0^1 d\tilde{y}\,\tilde{y}^{-1-\epsilon}$$

$$-\int_0^1 dx\, x^{-1-\epsilon}(1-x)^{2-\epsilon}\int_0^1 d\tilde{y}\,\tilde{y}^{-\epsilon} - 2\epsilon\int_0^1 dx\, x^{-\epsilon}(1-x)^{1-\epsilon}\int_0^1 d\tilde{y}\,\tilde{y}^{-\epsilon}\Bigg)$$

$$= -2(q^2)^{-\epsilon}\Gamma(1+\epsilon)(-1)^{-\epsilon}\Bigg(\frac{\Gamma(-\epsilon)\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)}\frac{\Gamma(-\epsilon)}{\Gamma(1-\epsilon)} - \frac{\Gamma(-\epsilon)\Gamma(3-\epsilon)}{\Gamma(3-2\epsilon)}\frac{\Gamma(1-\epsilon)}{\Gamma(2-\epsilon)}$$

$$-2\epsilon\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}\frac{\Gamma(1-\epsilon)}{\Gamma(2-\epsilon)}\Bigg)$$

$$= -2(q^2)^{-\epsilon}\Gamma(1+\epsilon)(-1)^{-\epsilon}\Bigg(\frac{\Gamma(-\epsilon)\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)}\frac{1}{-\epsilon} - \frac{\Gamma(-\epsilon)\Gamma(3-\epsilon)}{\Gamma(3-2\epsilon)}\frac{1}{1-\epsilon} - 2\epsilon\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}\frac{1}{1-\epsilon}\Bigg)$$

Add the results together, and clean up.

$$\delta F_1(q^2) = \frac{g^2 C_F \mu^{2\epsilon}}{(4\pi)^{2-\epsilon}}\Bigg\{2(1-\epsilon)^2\,\Gamma(\epsilon)(q^2)^{-\epsilon}(-1)^{-\epsilon}\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}\frac{1}{(1-\epsilon)}$$

$$-2(q^2)^{-\epsilon}\Gamma(1+\epsilon)(-1)^{-\epsilon}\Bigg(\frac{\Gamma(-\epsilon)\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)}\frac{1}{-\epsilon} - \frac{\Gamma(-\epsilon)\Gamma(3-\epsilon)}{\Gamma(3-2\epsilon)}\frac{1}{1-\epsilon} - 2\epsilon\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}\frac{1}{1-\epsilon}\Bigg)\Bigg\}$$

Factor $(q^2)^{-\epsilon}$ $(-1)^{-\epsilon}$, replace $\Gamma(1+\epsilon) = \epsilon\,\Gamma(\epsilon)$

$$= \frac{g^2 C_F \mu^{2\epsilon}(q^2)^{-\epsilon}}{(4\pi)^{2-\epsilon}}(-1)^{-\epsilon}\Bigg\{2(1-\epsilon)^2\,\Gamma(\epsilon)\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}\frac{1}{(1-\epsilon)}$$

$$-2\epsilon\,\Gamma(\epsilon)\Bigg(\frac{\Gamma(-\epsilon)\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)}\frac{1}{-\epsilon} - \frac{\Gamma(-\epsilon)\Gamma(3-\epsilon)}{\Gamma(3-2\epsilon)}\frac{1}{1-\epsilon} - 2\epsilon\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}\frac{1}{1-\epsilon}\Bigg)\Bigg\} \quad (*)$$

Back

Factor $\Gamma(\epsilon)$, rearrange leading coefficient.

$$= \frac{C_F}{4\pi}\left(\frac{g^2}{4\pi}\right)\left(\frac{4\pi\mu^2}{q^2}\right)^{\epsilon}\Gamma(\epsilon)(-1)^{-\epsilon}\Bigg\{2(1-\epsilon)^2\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}\frac{1}{1-\epsilon}$$

$$+\frac{2\,\Gamma(-\epsilon)\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)} + \frac{\Gamma(-\epsilon)\Gamma(3-\epsilon)}{\Gamma(3-2\epsilon)}\frac{2\epsilon}{1-\epsilon} + \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}\frac{4\epsilon^2}{1-\epsilon}\Bigg\}$$

Expand entire expression around $\epsilon = 0$ to $\mathcal{O}(\epsilon)^0$. Also, note $(-1)^{-\epsilon} \approx 1 - \pi i\epsilon - \frac{1}{2}\pi^2\epsilon^2 + \mathcal{O}(\epsilon)^3$.

$$\delta F_1(Q^2) = \frac{C_F\alpha_s}{4\pi}\Bigg(-\frac{2}{\epsilon^2} + \frac{1}{\epsilon}\bigg[-3 + 2\pi i + 2\gamma_E - 2\ln\left(\frac{4\pi\mu^2}{q^2}\right)\bigg] + \bigg[(-8+3\pi i) + (3-2\pi i)\gamma_E$$

$$+\frac{7\pi^2}{6} - \gamma_E^2 - (3 - 2\pi i - 2\gamma_E)\ln\left(\frac{4\pi\mu^2}{q^2}\right) - \underbrace{\ln^2\left(\frac{4\pi\mu^2}{q^2}\right)}_{\text{Sudakov dbl. log.}}\bigg] + \mathcal{O}(\epsilon)\Bigg) \quad (9)$$

use $\dfrac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} = (4\pi e^{-\gamma})^\epsilon\left(1 - \frac{\pi^2}{12}\epsilon^2\right) + \mathcal{O}(\epsilon)$