$\boxed{T \equiv \text{ in fundamental representation only.}}$

Trace Theorems for SU(N): The generators satisfy a number of trace theorems:

Starting with $\{T^a, T^b\} = \underset{\uparrow \text{ of } SU(N)}{\frac{1}{N}} \delta^{ab} + d^{abc} T^c$, $\quad d^{abc} \equiv$ totally symmetric constants (representation dependent)

we have:

$$\mathrm{Tr}[T^a T^b] = \mathrm{Tr}[-T^b T^a + \{T^a, T^b\}]$$

$\downarrow$ cyclic

$$= -\mathrm{Tr}[T^a T^b] + \mathrm{Tr}[\{T^a, T^b\}]$$

dimension of representation, = N for fundamental.

$$\mathrm{Tr}[T^a T^b] = \frac{1}{2}\mathrm{Tr}\left[\frac{1}{N}\delta^{ab}\mathbb{1} + d^{abc}T^c\right] = \delta^{ab}\frac{\cancel{N}}{2\cancel{N}} = \frac{1}{2}\delta^{ab}$$

$$\mathrm{Tr}[T^a T^b T^c] = \mathrm{Tr}[-T^b \overset{\text{commute}}{T^a T^c} + \{T^a, T^b\}T^c]$$

$$= \mathrm{Tr}\left[-T^b T^c T^a - T^b[T^a, T^c] + \{T^a, T^b\}T^c\right]$$

$\downarrow$ cyclic

$$= -\mathrm{Tr}[T^a T^b T^c] - \mathrm{Tr}[T^b[T^a, T^c]] + \mathrm{Tr}[\{T^a, T^b\}T^c]$$

$$\mathrm{Tr}[T^a T^b T^c] = \frac{1}{2}\left(-if^{acd}\,\mathrm{Tr}[T^b T^d] + \mathrm{Tr}\left[\left(\frac{1}{N}\delta^{ab}\mathbb{1} + d^{abd}T^d\right)T^c\right]\right)$$

$$= \frac{1}{2}\left(-if^{acd}\delta^{bd}\frac{n_r}{2N} + d^{abd}\,\mathrm{Tr}[T^d T^c]\right) \quad \text{since } \mathrm{Tr}[T^c] = 0.$$

$$= \frac{1}{2}\left(-if^{a\overset{\ominus}{cb}}\frac{n_r}{2N} + d^{abd}\delta^{dc}\frac{n_r}{2N}\right)$$

$$= \frac{1}{2}\left(if^{abc}\frac{n_r}{2N} + d^{abc}\frac{n_r}{2N}\right)$$

$$\mathrm{Tr}[T^a T^b T^c] = \frac{\cancel{n_r}}{4\cancel{N}}\left(if^{abc} + d^{abc}\right)$$

$\mathrm{Tr}[T^a T^b T^c T^d]$: anticommute T^a through 3 times, and use the cyclic property of trace to obtain

$$\mathrm{Tr}[T^a T^b T^c T^d] = \frac{1}{2}\mathrm{Tr}\left[T^b T^c\{T^a, T^d\} - T^b\{T^a, T^c\}T^d + \{T^a, T^b\}T^c T^d\right]$$

$$\mathrm{Tr}[T^aT^bT^cT^d] = \frac{1}{2}\,\mathrm{Tr}\Big[T^bT^c\Big(\frac{\delta^{ad}}{N} + d^{ade}T^e\Big) - T^b\Big(\frac{\delta^{ac}}{N} + d^{ace}T^e\Big)T^d + \Big(\frac{\delta^{ab}}{N} + d^{abe}T^e\Big)T^cT^d\Big]$$

$$= \frac{1}{2}\,\mathrm{Tr}\Big[\frac{1}{N}\Big(T^bT^c\delta^{ad} - T^b\delta^{ac}T^d + \delta^{ab}T^cT^d\Big) + d^{ade}T^bT^cT^e - d^{ace}T^bT^eT^d + d^{abe}T^eT^cT^d\Big]$$

$$= \frac{1}{2}\Big[\frac{n_r}{2N^2}\Big(\delta^{bc}\delta^{ad} - \delta^{bd}\delta^{ac} + \delta^{ab}\delta^{cd}\Big) + \frac{n_r}{4N}\Big(d^{ade}(if^{bce}+d^{bce}) - d^{ace}(if^{bed}+d^{bed}) + d^{abe}(if^{ecd}+d^{ecd})\Big)\Big]$$

$$\mathrm{Tr}[T^aT^bT^cT^d] = \frac{n_r}{4N^{\cancel{2}}}\Big(\delta^{bc}\delta^{ad} - \delta^{bd}\delta^{ac} + \delta^{ab}\delta^{cd}\Big)$$
$$+ \frac{\cancel{n_r}}{4\cancel{N}}\Big(d^{ade}d^{bce} - d^{ace}d^{bde} + d^{abe}d^{cde}\Big) + \frac{\cancel{n_r}}{4\cancel{N}}\, i\Big(d^{ade}f^{bce} + d^{ace}f^{bde} + d^{abe}f^{cde}$$

Contraction Identities for SU(N)

$T^a T^a = C_F \mathbb{1}$

Also useful: $i f^{abc} T^b T^c = -\frac{1}{2} C_A T^a$

$$T^a T^b T^a = T^b \underbrace{T^a T^a} + [T^a, T^b] T^a$$

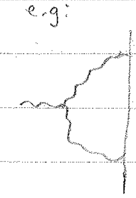
$$= C_F T^b + i f^{abc} T^c T^a$$

$$i f^{bca} T^c T^a$$

$$= C_F T^b + -\frac{1}{2} C_A T^b$$

$$= (C_F - \frac{1}{2} C_A) T^b .$$

e.g:



$$\det(U) = \exp(\mathrm{Tr}\{\ln U\}) = 1 + \mathrm{i}\,\delta\theta_a \mathrm{Tr}\{T^a\} + \cdots = 1 \quad \rightarrow \quad \mathrm{Tr}\{T^a\} = 0\,. \quad (A.25)$$

One sees that the generators are traceless hermitian matrices. If the factor i were not included in eqn (A.5), the generators would be anti-hermitian. Note that these constraints are sufficient to give a general description of how to build a set of generators for SU(N). The starting point is the set of Pauli matrices. The non-diagonal ones can be formed by putting a single 1 or an i into one position of the upper triangular matrix. Hermiticity then defines the entire matrix. This procedure already gives $N(N-1)$ traceless hermitian matrices, that is, one only needs to find another $N-1$ such matrices, to have all generators of SU(N). Those missing matrices can be chosen as diagonal matrices proportional to diag$(1, 1, \cdots, 1, -m, 0, \cdots, 0)$. Here the first m positions on the diagonal, $m = 1, \ldots, N-1$, are filled with unit elements. The next element is then set to $-m$ in order to have a traceless matrix, and the rest of the diagonal is filled with zeros. In the case of SU(3) the diagonal matrices constructed according to this scheme are diag$(1,-1,0)$ and diag$(1,1,-2)$. In general, for SU(N) we have $N-1$ generators which can be simultaneously diagonalized. In terms of physics this means that a quantum state with an SU(N) symmetry is characterized by $N-1$ quantum numbers.

In practical applications, such as the evaluation of Feynman diagrams, one is often faced with the problem of summing over representation matrices of a symmetry group. In many cases one has to deal with SU(N). Therefore, we list here a collection of identities valid for SU(N) (MacFarlane *et al.*, 1968), which are useful for performing such calculations. Note that the summation convention is used throughout.

Let T^a denote the generators for the fundamental respresentation of SU(N). The commutation relations are given by the totally antisymmetric structure constants f_{abc} of the group:

$$[T^a, T^b] = \mathrm{i}\, f^{abc} T^c \quad (A.26)$$

A corresponding relation also exists for the anticommutator

$$\{T^a, T^b\} = \frac{1}{N}\delta_{ab} I_N + d^{abc} T^c\,, \quad (A.27)$$

with a totally symmetric tensor d^{abc}. Here, I_N is the N-dimensional unit matrix. For SU(2) one has $f^{abc} = \varepsilon^{abc}$ and $d^{abc} = 0$. Some useful formulae for the T^a are

$$T^a T^b = \frac{1}{2}\left[\frac{1}{N}\delta_{ab} I_N + (d^{abc} + i f^{abc}) T^c\right]\,, \quad (A.28)$$

$$\boxed{T^a_{ij} T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right)}\,, \quad (A.29)$$

$$\mathrm{Tr}\{T^a\} = 0\,, \quad (A.30)$$

$$\mathrm{Tr}\{T^a T^b\} = \frac{1}{2}\delta_{ab}\,, \quad (A.31)$$

$$\mathrm{Tr}\{T^a T^b T^c\} = \frac{1}{4}(d^{abc} + i f^{abc})\,, \quad (A.32)$$

$$\mathrm{Tr}\{T^a T^b T^a T^c\} = -\frac{1}{4N}\delta_{bc}\,. \quad (A.33)$$

In addition one has the Jacobi identities

$$f^{abe} f^{ecd} + f^{cbe} f^{aed} + f^{dbe} f^{ace} = 0\,, \quad (A.34)$$

$$f^{abe} d^{ecd} + f^{cbe} d^{aed} + f^{dbe} d^{ace} = 0\,, \quad (A.35)$$

which can be written in a compact form by introducing the (N^2-1)-dimensional matrices $(F^a)_{bc} = -\mathrm{i}\, f^{abc}$ and $(D^a)_{bc} = d^{abc}$ as

$$[F^a, F^b] = \mathrm{i}\, f^{abc} F^c \quad \text{and} \quad [F^a, D^b] = \mathrm{i}\, f^{abc} D^c\,. \quad (A.36)$$

Further useful identities involving products and traces are

$$f^{abe} f^{cde} = \frac{2}{N}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + (d^{ace} d^{bde} - d^{bce} d^{ade})\,, \quad (A.37)$$

$$f^{abb} = \mathrm{Tr}\{F^a\} = 0\,, \quad (A.38)$$

$$d^{abb} = \mathrm{Tr}\{D^a\} = 0\,, \quad (A.39)$$

$$f^{acd} f^{bcd} = \mathrm{Tr}\{F^a F^b\} = N\delta_{ab}\,, \quad (A.40)$$

$$f^{acd} d^{bcd} = 0\,, \quad (A.41)$$

$$d^{acd} d^{bcd} = \mathrm{Tr}\{D^a D^b\} = \frac{N^2-4}{N}\delta_{ab}\,, \quad (A.42)$$

$$F^a F^a = N I_{N^2-1}\,, \quad (A.43)$$

$$\mathrm{Tr}\{F^a F^b F^c\} = i\frac{N}{2} f^{abc}\,, \quad (A.44)$$

$$\mathrm{Tr}\{D^a F^b F^c\} = \frac{N}{2} d^{abc}\,, \quad (A.45)$$

$$\mathrm{Tr}\{D^a D^b F^c\} = i\frac{N^2-4}{2N} f^{abc}\,, \quad (A.46)$$

$$\mathrm{Tr}\{D^a D^b D^c\} = \frac{N^2-12}{2N} d^{abc}\,. \quad (A.47)$$

A.3 Colour factors

Colour factors are defined as the eigenvalues of the quadratic Casimir operator for a given representation, with the normalization usually fixed through the Dynkin index T_F of the fundamental representation. Below, we give a table of the ratios T_F/C_F and C_A/C_F, with C_F and C_A the colour factors of the fundamental and the adjoint representation, respectively, for all semi-simple Lie groups.