Fadeev – Popov – (DeWitt) Quantization of Gauge Theories

Want to evaluate $Z = \int \mathcal{D}A \, e^{i\int d^4x \, -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}}$

Write:
$$S_{Maxwell} = \int d^4x \, -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}$$
$$= \int d^4x \, -\frac{1}{2} \partial_\mu A^a_\nu \left(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a} \right) + \text{"}A^3\text{"} + \text{"}A^4\text{"}$$
$$= \int d^4x \, +\frac{1}{2} A^a_\nu \left(\partial_\mu \partial^\mu A^{\nu a} - \partial_\mu \partial^\nu A^{\mu a} \right) + \text{Surface terms}\Big|_\infty$$
$$= \int d^4x \, +\frac{1}{2} A^a_\mu \left(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu \right) A^a_\nu + \text{"}A^3\text{"} + \text{"}A^4\text{"}$$

Performing the Gaussian path integration is problematic, since the operator $\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu$ has no inverse. – Has a vanishing e-value:

$$\left(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu \right) \underbrace{\partial_\mu f(x)}_{\text{e-function}} = \left(\partial^2 \partial^\nu - \partial^2 \partial^\nu \right) f(x) = 0$$

The root of the problem is that the functional integration $Z = \int \mathcal{D}A \, e^{iS_{Maxwell}}$ is ill-defined.

– It contains a redundant integral

→ Easiest to see in discrete form, as an n-dimensional integral:

$$Z = \int \mathcal{D}A \, e^{iS_{Maxwell}} \simeq \int \mathcal{D}A \, e^{i\int d^4x \, \frac{1}{2} A \,\mathcal{O}\, A + \ldots}$$

$$\xrightarrow{\text{discretize}} \underbrace{\int_{-\infty}^{\infty} dA_1 \, dA_2 \ldots dA_n}_{} \exp\left[\frac{i}{2} \sum_{i,j} A_i \, (\text{operator})_{ij} \, A_j \right]$$

each integral associated with field variable at one spacetime point, one Lorentz component, one color component, ...

$$\exp\left[\frac{i}{2} \left(A_1 \ldots A_n \right) \left(\text{operator} \right) \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix} \right]$$

The idea is to diagonalize this matrix.

Introduce special orthogonal rotation matrix, $\mathbb{R}$.

$\det \mathbb{R} = 1$ $\qquad$ $\mathbb{R}^T \mathbb{R} = 1$

$$Z = \int dA_1 \dots dA_n \, \exp\left[\frac{i}{2} (A_1 \dots A_n) \overbrace{\left(\mathbb{R}^T \right) \left(\mathbb{R} \right)}^{\text{insert } \mathbb{1}} \!\!\!\! \underbrace{\left(\text{operator} \right) \left(\mathbb{R}^T \right)}_{} \!\!\!\! \overbrace{\left(\mathbb{R} \right)}^{\text{insert } \mathbb{1}} \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix} \right]$$

$$\mathbb{R} \left(\text{operator} \right) \mathbb{R}^T = \mathbb{D} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \quad \text{diagonal matrix.}$$

$$= \int dA_1 \dots dA_n \, \exp\left[\frac{i}{2} (A_1 \dots A_n) \left(\mathbb{R}^T \right) \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \left(\mathbb{R} \right) \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix} \right]$$

Change integration variables: $A_i' = \mathbb{R}_{ij} A_j$

$$= \int dA_1' \dots dA_n' \, \exp\left[\frac{i}{2} (A_1' \dots A_n') \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} A_1' \\ \vdots \\ A_n' \end{pmatrix} \right]$$

$$= \int dA_1' \, e^{\frac{i}{2} \lambda_1 A_1'^2} \times \int dA_2' \, e^{\frac{i}{2} \lambda_2 A_2'^2} \times \dots \times \int dA_n' \, e^{\frac{i}{2} \lambda_n A_n'^2}$$

For every e-value, λ, that vanishes, there will be a spurious, divergent integral, since there will be no Gaussian damping.

For example: If $\lambda_1 = 0$, we have

$$Z = \underbrace{\int_{-\infty}^{\infty} dA_1' \, (1)}_{\text{divergent } = \infty} \times \int dA_2' \, e^{\frac{i}{2} \lambda_2 A_2'^2} \times \dots \times \int dA_n' \, e^{\frac{i}{2} \lambda_n A_n'^2} = \infty \, !$$

But, this should not render the theory pathological since physical quantities follow from correlation functions, that a ratios of path integrals.

$$\langle \mathcal{O} \rangle = \frac{\int dA_1' \dots dA_n' \, \mathcal{O} \, \exp\left[\frac{i}{2} (\lambda_1 A_1^2 + \dots \lambda_n A_n^2) \right]}{\int dA_1' \dots dA_n' \, \exp\left[\frac{i}{2} (\lambda_1 A_1^2 + \dots \lambda_n A_n^2) \right]}$$

If $\lambda_1 = 0$ (as in our example), we can factor out the redundant A_1' integration, provided $\mathcal{O}$ is indep of A_1' $\leftarrow$ statement about gauge invariant observables

$$\langle \mathcal{O} \rangle = \frac{\cancel{\int dA_1' (1)} \int dA_2' \dots dA_n' \, \mathcal{O} \, \exp\left[\frac{i}{2} (\dots \lambda_n A_n^2) \right]}{\cancel{\int dA_1' (1)} \int dA_2' \dots dA_n' \, \exp\left[\frac{i}{2} (\dots \lambda_n A_n^2) \right]} = \text{finite}$$

Gauge Fixing

$$Z = \int \mathcal{D}A\, e^{iS[A]} = \left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A\, e^{iS[A]} \underbrace{\det\left(\frac{\delta G^a[A'(x)]}{\delta\alpha^b(y)}\right)}_{①} \underbrace{\delta(G^a[A])}_{②}$$

→ choose generalized Lorentz gauge condition:

$$G^a[A(x)] = \underbrace{\partial^\mu A^a_\mu(x)}_{\mathcal{F} \equiv \text{Fixing condition (vanishes for physical states)}} - \omega^a(x)$$

$\omega^a(x)$: Arbitrary "classical" function dependent on x.

Then

$$G^a[A'(x)] = \partial^\mu\left(A^a_\mu - \frac{1}{g}\partial_\mu\alpha^a - f^{abc}\alpha^b A^c_\mu\right) - \omega^a(x)$$

$$= \partial^\mu A^a_\mu - \frac{1}{g}\partial^\mu\partial_\mu\alpha^a - f^{abc}(\partial^\mu\alpha^b)A^c_\mu - f^{abc}\alpha^b\partial^\mu A^c_\mu - \omega^a(x)$$

At this point, we are done fixing the gauge. However, in order to perform calculations, the δ-functional & determinant need to be lifted into the exponent.

① $$\frac{\delta G^a[A'(x)]}{\delta\alpha^b(y)} = \frac{\partial G}{\partial\alpha^b}\delta(x-y) + \frac{\partial G}{\partial(\partial^\mu\alpha^b)}\partial^\mu(\delta(x-y)\cdot) + \frac{\partial G}{\partial(\partial_\mu\partial_\nu\alpha^b)}\partial_\mu\partial_\nu(\delta(x-y)\cdot)$$

$$= -f^{abc}\partial^\mu A^c_\mu\,\delta(x-y) - f^{abc}A^c_\mu\underbrace{\partial^\mu}_{\text{hungry derivative}}(\delta(x-y)\cdot) - \frac{1}{g}\delta^{ab}g^{\mu\nu}\underbrace{\partial_\mu\partial_\nu}_{\text{hungry derivatives}}(\delta(x-y)\cdot)$$

$$= -f^{abc}\partial^\mu(A^c_\mu\delta(x-y)\cdot) - \frac{1}{g}\delta^{ab}\partial^\mu\partial_\mu(\delta(x-y)\cdot)$$

$$= -\frac{1}{g}\partial^\mu\left[\delta^{ab}\partial_\mu + gf^{abc}A^c_\mu\right]\delta(x-y)\cdot$$

$$\equiv -\frac{1}{g}\partial^\mu D^{ab}_\mu\,\delta(x-y)\cdot$$

Then $$\det\left(\frac{\delta G^a[A'(x)]}{\delta\alpha^b(y)}\right) = \det\left(-\frac{1}{g}\partial^\mu D^{ab}_\mu\delta(x-y)\cdot\right)$$

$$= \int \mathcal{D}\eta\,\mathcal{D}\bar\eta\; e^{i\int d^4x\,d^4y\left[\bar\eta^a(x)\underbrace{\tfrac{-1}{g}}_{\text{absorb into normalization for }\eta}\partial^\mu D^{ab}_\mu(\delta(x-y)\eta^b(y))\right]}$$

$$= \int \mathcal{D}\eta\,\mathcal{D}\bar\eta\; e^{i\int d^4x\,d^4y\left[-\bar\eta^a(x)\partial^\mu D^{ab}_\mu\delta(x-y)\eta^b(y)\right]}$$

η & $\bar\eta$ ≡ (independent) Grassman valued fields (Fadeev-Popov ghosts).

Simplify exponent:

$$= i\int d^4x \int d^4y \, -\bar{\eta}^a(x)\; \partial^\mu D_\mu^{ab}\left(\delta(x-y)\,\eta^b(y)\right)$$

$$= i\int d^4x \int d^4y \, -\bar{\eta}^a(x)\; \partial^\mu\left(\partial_\mu \delta^{ab} + g f^{abc} A_\mu^c\right)\delta(x-y)\,\eta^b(y)$$

$$= i\int d^4x \int d^4y \left[-\bar{\eta}^a(x)\,\delta^{ab}\,\underbrace{\partial^\mu\partial_\mu}\left(\delta(x-y)\,\eta^b(y)\right) - \bar{\eta}^a(x)\, g f^{abc}\,\partial^\mu\left(A_\mu^c(x)\,\delta(x-y)\,\eta^b(y)\right)\right]$$

- Integrate by parts twice, exposing $\delta(x-y)$
- Integrate over y, fixing $y \to x$
- Integrate by parts once, putting ∂ on η.
  $\Rightarrow$ (gained overall -1)

- Integrate by parts once
- Integrate over y fixing $y \to x$
  $\Rightarrow$ (gained overall -1)

$$= \quad i\int d^4x \left[\partial_\mu \bar{\eta}^a\; \partial^\mu \eta^a \;+ g f^{abc}\,(\partial^\mu \eta^a)\,\eta^b A_\mu^c\right]$$

(2) Now work on δ-functional:

$$\delta(G[A]) \;=\; \delta\left(\partial^\mu A_\mu^a(x) - \omega^a(x)\right)$$

The functional integral, Z, does not depend on the form of $\omega^a(x)$ (how could it? It was introduced by multiplying by $1 = \int \mathcal{D}\alpha\, \Delta_G(G(A))$). I can freely divide the integral into two halves — each involving a different form for $\omega^a(x)$:

$$Z = \frac{1}{2}\int \mathcal{D}\alpha \int \mathcal{D}A\; e^{iS[A]}\; \det\left(\frac{\delta G^a[A'(x)]}{\delta \alpha(y)}\right)\,\delta\left(\partial^\mu A_\mu^a - \omega_1^a(x)\right)$$

$$+\; \frac{1}{2}\int \mathcal{D}\alpha \int \mathcal{D}A\; e^{iS[A]}\; \det\left(\frac{\delta G^a[A'(x)]}{\delta \alpha(y)}\right)\,\delta\left(\partial^\mu A_\mu^a - \omega_2^a(x)\right).$$

Similarly, I can "continuously" divide up the functional integral, with each "term" involving an "infinitessimally" different form for $\omega^a(x)$, and the "continuously" add them back together as long as the overall normalization factor is unchanged.

$$Z = \int \mathcal{D}\omega^a\; \underbrace{e^{-i\int d^4x \frac{(\omega^a(x))^2}{2\xi}}}_{\text{Gaussian normalization}} \int \mathcal{D}\alpha \int \mathcal{D}A\; e^{iS[A]}\; \det\left(\frac{\delta G^a[A'(x)]}{\delta \alpha^b(y)}\right)\,\delta\left(\partial^\mu A_\mu^a - \omega^a(x)\right)$$

Explores the space of all $\omega^a(x)$.

Integrate over $\omega^a(x)$, using δ-functional, fixing $\omega^a(x) \to \partial^\mu A^a_\mu(x)$

$$Z = \int \mathcal{D}\alpha \int \mathcal{D}A \, e^{iS[A]} \, e^{-i\int d^4x \frac{1}{2\xi}(\partial.A)^2} \det\left(\frac{\delta G^a[A'(x)]}{\delta \alpha^b(y)}\right)$$

$$= \underbrace{\int \mathcal{D}\alpha}_{\substack{\| \\ \left(\text{volume of gauge group}\right)\left(\text{volume of space}\right)}} \int \mathcal{D}A \, \mathcal{D}\eta \, \mathcal{D}\bar\eta \; e^{iS[A] - i\int d^4x \frac{1}{2\xi}(\partial.A)^2 + \partial_\mu \bar\eta \partial^\mu \eta + g f^{abc}(\partial^\mu \bar\eta^a)\eta^b A^c_\mu}$$

← This infinite factor gets divided out.

The resulting gauge-fixed Lagrangian is:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a} - \frac{1}{2\xi}(\partial.A)^2 + \partial_\mu \bar\eta \partial^\mu \eta + g f^{abc}(\partial^\mu \bar\eta^a)\eta^b A^c_\mu$$

$$\boxed{\begin{aligned} &\equiv \frac{1}{2} A^a_\mu (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A^a_\nu + \frac{1}{2\xi} A^a_\mu \partial^\mu \partial^\nu A^a_\nu + \partial_\mu \bar\eta \partial^\mu \eta \\ &\quad + g f^{abc} \partial_\mu A^a_\nu A^{\mu b} A^{\nu c} - \frac{g^2}{4} f^{abc} f^{cde} A^a_\mu A^b_\nu A^{\mu d} A^{\nu e} + g f^{abc} (\partial^\mu \bar\eta^a) \eta^b A^c_\mu . \end{aligned}}$$

For a general gauge fixing condition: $G^a[A] = \mathcal{F}^a(x) - \omega^a(x)$,
we have

↖ Arbitrary gauge fixing condition.

$$Z = \left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A \, e^{iS[A]} \det\left(\frac{\delta \mathcal{F}'^a(x)}{\delta \alpha^b(y)}\right) \delta\left(\mathcal{F}^a(x) - \omega^a(x)\right)$$

$$\vdots$$

$$\Rightarrow \mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} - \frac{1}{2\xi}(\mathcal{F}^a)^2 + \int d^4y \, \bar\eta^a(x) \left[\frac{\delta \mathcal{F}'^a(x)}{\delta \alpha^c(y)}\right] \eta^c(y)$$

↑ differential operators in here are "hungry" derivatives — act on everything to the right.