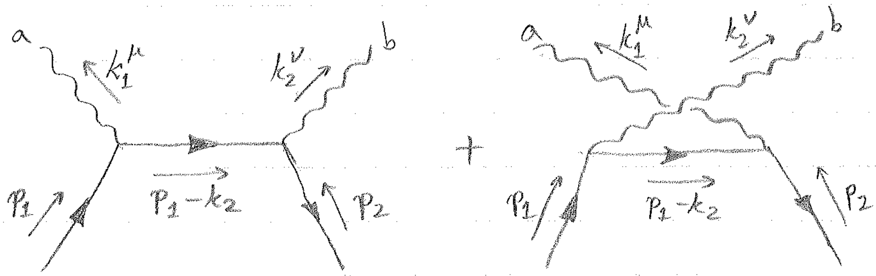
Ward Identity in a Non-Abelian Gauge Theory

Check the Ward Identity for $q\bar{q} \to gg$ to leading order in coupling constant, g^2.



$$p_1 + p_2 = k_1 + k_2$$
$$p_1 - k_2 = k_1 - p_2$$

$$i\mathcal{M}^{\mu\nu}_{ab} = \bar{v}(p_2)\Big[(-ig\gamma^\nu T^b)\frac{i}{\not p_1 - \not k_1 - m}(-ig\gamma^\mu T^a)$$
$$+(-ig\gamma^\mu T^a)\frac{i}{\not p_1 - \not k_2 - m}(-ig\gamma^\nu T^b)\Big]u(p_1)\,\epsilon^*_\mu(k_1)\,\epsilon^*_\nu(k_2)$$

To check Ward identity, replace $\epsilon^*_\mu(k_1) \to k_{1\mu}$

$$ik_{1\mu}\mathcal{M}^\mu = -ig^2\,\bar{v}(p_2)\Big[\gamma^\nu T^b \frac{1}{\not p - \not k_1 - m}\gamma^\mu T^a + \gamma^\mu T^a \frac{1}{\underbrace{\not p_2 - \not k_2}_{\text{replace with } k_1 - p_2} - m}\gamma^\nu T^b\Big]u(p_1)\,k_{1\mu}\,\epsilon^*_\nu(k_2)$$

$$= -ig^2\,\bar{v}(p_2)\Big[\gamma^\nu T^b \frac{1}{\not p - \not k_1 - m}\not k_1 T^a + \not k_1 T^a \frac{1}{k_1 - p_2 - m}\gamma^\nu T^b\Big]u(p_1)\,\epsilon^*_\nu(k_2)$$

rewrite: $k_1 - (p_1 - m) + \underbrace{(p_1 - m)}$ $\quad k_1 - (p_2 + m) + \underbrace{(p_2 + m)}$

These pieces vanish by Dirac eqn

$$= -ig^2\,\bar{v}(p_2)\Big[\gamma^\nu T^b \frac{\not k_1 - (\not p_1 - m)}{\not p_1 - \not k_1 - m} T^a + T^a \frac{\not k_1 - (\not p_2 + m)}{\not k_1 - \not p_2 - m}\gamma^\nu T^b\Big]u(p_1)\,\epsilon^*_\nu(k_2)$$

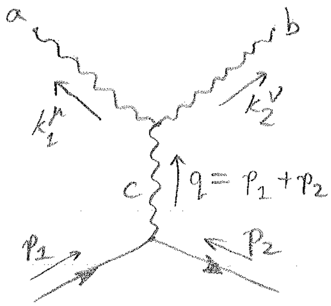
$$= -ig^2\,\bar{v}(p_2)\Big[-\gamma^\nu T^b T^a + T^a \gamma^\nu T^b\Big]u(p_1)\,\epsilon^*_\nu(k_2)$$

$$= -ig^2\,\bar{v}(p_2)\,\gamma^\nu\,[T^a, T^b]\,u(p_1)\,\epsilon^*_\nu(k_2)$$

$[T^a, T^b] = if^{abc}T^c$

$$= +g^2\,\bar{v}(p_2)\left(\gamma^\nu f^{abc}\,T^c\right)u(p_1)\,\epsilon^*_\nu(k_2)$$

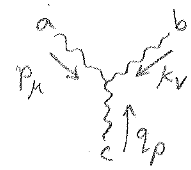
Non-zero, but there is a third diagram to consider due to the non-abelian nature of the theory.



$q = k_1 + k_2$

$\Rightarrow k_1 = q - k_2$

Relevant Feynman rule:



$$= g f^{abc}\left(g_{\mu\nu}(k-p)_\rho + g_{\nu\rho}(q-k)_\mu + g_{\rho\mu}(p-q)_\nu\right)$$

To apply it to our case,

$$p \to -k_1, \quad k \to -k_2, \quad \rho \to \sigma.$$

$$i\mathcal{M} = \bar{v}(p_2)(-ig\gamma^\rho T^c)u(p_1)\frac{-ig_{\rho\sigma}}{q^2} g f^{abc}\left[g^{\mu\nu}(-k_2+k_1)^\sigma + g^{\nu\sigma}(q+k_2)^\mu + g^{\sigma\mu}(-k_1-q)^\nu\right]\underbrace{\epsilon^*_\mu(k_1)}_{\substack{\text{Replace with } k_{1\mu}\\ \text{to check Ward identity}}}\epsilon^*_\nu(k_2)$$

Eliminate k_1 in favor of q & k_2 using $k_1 = q - k_2$.

$$i k_{1\mu}\mathcal{M}^\mu = \bar{v}(p_2)\ldots g f^{abc}\left[g^{\mu\nu}(-k_2+q-k_2)^\sigma + g^{\nu\sigma}(k_2+q)^\mu + g^{\sigma\mu}(-q+k_2-q)^\nu\right](q-k_2)_\mu\,\epsilon^*_\nu(k_2)$$

$$= \bar{v}(p_2)\ldots g f^{abc}\left[(q-k_2)^\nu(q-2k_2)^\sigma + g^{\nu\sigma}(k_2+q)^\mu(q-k_2)_\mu + (q-k_2)^\sigma(-2q+k_2)^\nu\right]\epsilon^*_\nu(k_2)$$

$$= \bar{v}(p_2)\ldots g f^{abc}\Big[\left(\cancel{q^\nu q^\sigma} - \cancel{2q^\nu k_2^\sigma} - \cancel{k_2^\nu q^\sigma} + \cancel{2}k_2^\nu k_2^\sigma\right) + g^{\nu\sigma}(q^2-k_2^2)$$
$$+ \left(-\cancel{2}q^\sigma q^\nu + \cancel{q^\sigma k_2^\nu} + \cancel{2k_2^\sigma q^\nu} - \cancel{k_2^\sigma k_2^\nu}\right)\Big]\epsilon^*_\nu(k_2)$$

lower index

$$= \bar{v}(p_2)(-ig\gamma^\rho T^c)u(p_1)\frac{-ig_{\rho\sigma}}{q^2} g f^{abc}\left[k_2^\nu k_2^\sigma + g^{\nu\sigma}(q^2-k_2^2) - q^\sigma q^\nu\right]\epsilon^*_\nu(k_2)$$

If other external gluon is on-shell, it must be transversely polarized. i.e. if $k_2^2 = 0$, then $k_2^\nu \epsilon^*_\nu(k_2) = 0$. } Make this assumption

$$= -g^2\bar{v}(p_2)(\gamma_\sigma T^c)u(p_1)\frac{1}{q^2} f^{abc}\left[g^{\nu\sigma}q^2 - q^\sigma q^\nu\right]\epsilon^*_\nu(k_2)$$

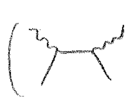
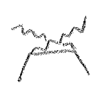
All-important minus sign!

$$\not{q} = \not{p}_1 + \not{p}_2$$
$$= (\not{p}_1 - m) + (\not{p}_2 + m)$$

Vanishes by Dirac eqn $(\not{p}_1 - m)u(p_1) = 0, \quad \bar{v}(p_2)(\not{p}_2 + m) = 0$

$$= -g^2\bar{v}(p_2)(\gamma_\sigma T^c)u(p_1)\frac{1}{\cancel{q^2}} f^{abc}\left(g^{\nu\sigma}\cancel{q^2}\right)\epsilon^*_\nu(k_2)$$

$$= -g^2\bar{v}(p_2)\left(\gamma^\nu f^{abc}T^c\right)u(p_1)\,\epsilon^*_\nu(k_2)$$

Cancels with ( + ), preserving the Ward identity, $k_\mu\mathcal{M}^\mu = 0$.

all ext. part. on-shell.