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Renormalization of QCD

$$\mathcal{L}_R = \sum_f \bar{\psi}_f (i\not{\partial} - m_{B,f})\psi_f + \frac{1}{2} A^a_\mu (g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu) A^a_\nu - \frac{1}{2\xi_B}(\partial . A)^2$$
$$+ \partial_\mu \bar{\eta}^a \partial^\mu \eta^a - g_B A^a_\mu \bar{\psi}_f \gamma^\mu T^a \psi_f$$
$$+ g_B f^{abc} (\partial_\mu \bar{\eta}^a) \eta^b A^{\mu c} + g_B f^{abc} (\partial_\mu A^a_\nu) A^{\mu b} A^{\nu c}$$
$$- \frac{g_B^2}{4} f^{abc} f^{cde} A^a_\mu A^b_\nu A^{\mu d} A^{\nu e} \qquad \text{(all fields bare)}$$

Renormalize field strengths:

$$\psi_{\text{Bare}} = \sqrt{Z_\psi}\, \psi, \quad A^\mu_{\text{Bare}} = \sqrt{Z_A}\, A^\mu, \quad \eta^a_{\text{Bare}} = \sqrt{Z_\eta}\, \eta^a$$

$$\mathcal{L}_R = \sum_f Z_{\psi_f} \bar{\psi}_f (i\not{\partial} - m_{B,f})\psi_f + \frac{1}{2} Z_A A^a_\mu (g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu) A^a_\nu - \frac{1}{2\xi_B} Z_A (\partial . A)^2$$
$$+ Z_\eta \partial_\mu \bar{\eta}^a \partial^\mu \eta^a - \sum_f g_B Z_A^{1/2} Z_{\psi_f} A^a_\mu \bar{\psi}_f \gamma^\mu T^a \psi_f$$
$$+ g_B Z_A^{1/2} Z_\eta f^{abc} (\partial_\mu \bar{\eta}^a) \eta^b A^{\mu c} + g_B Z_A^{3/2} f^{abc} (\partial_\mu A^a_\nu) A^{\mu b} A^{\nu c}$$
$$- \frac{g_B^2}{4} Z_A^2 f^{abc} f^{cde} A^a_\mu A^b_\nu A^{\mu d} A^{\nu e}$$

Now renormalize composite operators: (in dim. reg. $d = 4 - 2\epsilon$)

$$\mathcal{L}_R = \sum_f Z_{\psi_f} \bar{\psi}_f (i\not{\partial})\psi_f - \sum_f Z_{\psi\psi_f} m_{R,f} \bar{\psi}_f \psi_f + \frac{1}{2} Z_A A^a_\mu (\dots) A^a_\nu - \frac{1}{2\xi_R} Z_{(\partial A)^2} (\partial . A)^2$$
$$+ Z_\eta \partial_\mu \bar{\eta}^a \partial^\mu \eta^a - \sum_f g_R \mu^\epsilon Z_{A\psi\psi_f} A^a_\mu \bar{\psi}_f \gamma^\mu T^a \psi_f$$
$$+ g_R \mu^\epsilon Z_{\eta\eta A} f^{abc} (\partial_\mu \bar{\eta}^a) \eta^b A^{\mu c} + g_R \mu^\epsilon Z_{A^3} f^{abc} (\partial_\mu A^a_\nu) A^{\mu b} A^{\nu c}$$
$$- \frac{g_R^2}{4} \mu^{2\epsilon} Z_{A^4} f^{abc} f^{cde} A^a_\mu A^b_\nu A^{\mu c} A^{\nu d}$$

where:

$$Z_{\psi\psi_f} m_{R,f} = m_{B,f} Z_{\psi_f} \qquad g_R \mu^\epsilon Z_{\eta\eta A} = g_B Z_A^{1/2} Z_\eta$$

$$\frac{1}{2\xi_R} Z_{(\partial A)^2} = \frac{1}{2\xi_B} Z_A \qquad g_R \mu^\epsilon Z_{A^3} = g_B Z_A^{3/2}$$

$$g_R \mu^\epsilon Z_{A\psi\psi_f} = g_B Z_A^{1/2} Z_{\psi_f} \qquad g_R^2 \mu^{2\epsilon} Z_{A^4} = g_B Z_A^2$$

(useful to define: $Z_{m_f} = \dfrac{Z_{\psi\psi_f}}{Z_{\psi_f}} = \dfrac{m_{B,f}}{m_{R,f}}$)

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For gauge invariance to hold, we must have:

Universal coupling constant renormalization factor.

$$\frac{g_B}{g_R\mu^\epsilon} = \frac{Z_{A\psi\psi_f}}{Z_A^{1/2}Z_{\psi_f}} = \frac{Z_{\eta\eta A}}{Z_A^{1/2}Z_\eta} = \frac{Z_{A^3}}{Z_A^{3/2}} = \sqrt{\frac{Z_{A^4}}{Z_A^2}} \equiv Z_g$$

(This is ensured by the Taylor-Slavnov identities).

NOTE: The relationship among the various Z-factors is non-trivial. Equality among the bare coupling constants do not at all imply equality of the renormalized ones at the full quantum level. Such relationships are a consequence of an underlying symmetry.

To facilitate the renormalization program in the context of perturbation theory, separate the renormalized Lagrangian into two parts — a restored Lagrangian, and a collection of counterterms.

$$\begin{aligned}
\mathcal{L}_R = {} & \sum_f \bar\psi_f(i\partial\!\!\!/ - m_f)\psi_f + \tfrac{1}{2}A^a_\mu(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A^a_\nu - \frac{1}{2\xi}(\partial\cdot A)^2 \\
& + \partial_\mu\bar\eta^a\partial^\mu\eta^a - \sum_f g_R\mu^\epsilon A^a_\mu\bar\psi\gamma^\mu T^a\psi_f \\
& + g_R\mu^\epsilon f^{abc}(\partial_\mu\bar\eta^a)\eta^b A^{\mu c} + g_R\mu^\epsilon f^{abc}(\partial_\mu A^a_\nu)A^{\mu b}A^{\nu c} \\
& - \frac{g_R^2}{4}\mu^{2\epsilon} f^{abc}f^{cde}A^a_\mu A^b_\nu A^{\mu d}A^{\nu e}
\end{aligned}$$

Counterterm Lagrangian

$$\left\{\begin{aligned}
& + \sum_f \bar\psi_f(\delta_{\psi_f} i\partial\!\!\!/ - \delta_{m_f})\psi_f + \tfrac{1}{2}\delta_A A^a_\mu(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A^a_\nu - \tfrac{1}{2}\delta_\xi(\partial\cdot A)^2 \\
& + \delta_\eta\,\partial_\mu\bar\eta^a\partial^\mu\eta^a - \sum_f \delta_{A\psi\psi_f}\mu^\epsilon A^a_\mu\bar\psi_f\gamma^\mu T^a\psi_f \\
& + \delta_{\eta\eta A}\mu^\epsilon f^{abc}(\partial_\mu\bar\eta^a)\eta^b A^{\mu c} + \delta_{A^3}\mu^\epsilon f^{abc}(\partial_\mu A^a_\nu)A^{\mu b}A^{\nu c} \\
& - \frac{\delta_{A^4}\mu^{2\epsilon}}{4} f^{abc}f^{cde}A^a_\mu A^b_\nu A^{\mu d}A^{\nu e}
\end{aligned}\right.$$

where

$$Z_{\psi_f} = 1 + \delta_{\psi_f}, \qquad Z_{\psi\psi_f} = 1 + \frac{\delta_m}{m_R} \qquad Z_{A^3} = 1 + \frac{\delta_{A^3}}{g_R}$$

$$Z_A = 1 + \delta_A, \qquad Z_{A\psi\psi_f} = 1 + \frac{\delta_{A\psi\psi_f}}{g_R} \qquad Z_{A^4} = 1 + \frac{\delta_{A^4}}{g_R^2}$$

$$Z_\eta = 1 + \delta_\eta, \qquad Z_{\eta\eta A} = 1 + \frac{\delta_{\eta\eta A}}{g_R}$$