Feynman–Hellmann theorem for stationary states

Let $|\psi\rangle$ be a bound state of $\hat{H}$ with energy E satisfying:

$$\hat{H}\,|\psi\rangle = E\,|\psi\rangle.$$

If $\hat{H} \equiv \hat{H}(\lambda)$ ANALYTIC dependence on parameter. (not near singular points)

then $E \equiv E(\lambda)$ and $|\psi\rangle \equiv |\psi(\lambda)\rangle$.

Start with matrix element of Schrödinger equation:

$$\langle\psi|\hat{H}|\psi\rangle = E\,\langle\psi|\psi\rangle$$

Differentiate:

$$\frac{dE}{d\lambda}\cancel{\langle\psi|\psi\rangle} = \frac{d}{d\lambda}\Big(\langle\psi|\hat{H}|\psi\rangle\Big)$$

$$= \left(\frac{d}{d\lambda}\langle\psi|\right)\hat{H}|\psi\rangle + \langle\psi|\frac{d\hat{H}}{d\lambda}|\psi\rangle + \langle\psi|\hat{H}\left(\frac{d}{d\lambda}|\psi\rangle\right)$$

$$= E\left[\left(\frac{d}{d\lambda}\langle\psi|\right)|\psi\rangle + \langle\psi|\left(\frac{d}{d\lambda}|\psi\rangle\right)\right] + \langle\psi|\frac{d\hat{H}}{d\lambda}|\psi\rangle$$

$$= E\,\underbrace{\frac{d}{d\lambda}\Big(\underbrace{\langle\psi|\psi\rangle}_{1}\Big)}_{0} + \langle\psi|\frac{d\hat{H}}{d\lambda}|\psi\rangle$$

$$\boxed{\frac{dE}{d\lambda} = \langle\psi|\frac{d\hat{H}}{d\lambda}|\psi\rangle}$$

Feynman-Hellmann theorem.

Power of theorem: expectation value in RHS is evaluated at $|\psi\rangle$ for fixed λ.

Exact parametric dependence of λ on $|\psi\rangle$ need not be known.

(Numerically obtained $|\psi\rangle$ would suffice!)

Caveat: Must be at an analytic point in complex λ-plane!

Can't study interacting spectrum starting from free theory.