Pauli's theory of spin

Spin operators: $\vec{S} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$

If quantization axis $\equiv \vec{e}$,

then $\vec{S}\cdot\vec{e}\,|e,+\rangle = \frac{\hbar}{2}|e,+\rangle$

$\vec{S}\cdot\vec{e}\,|e,-\rangle = -\frac{\hbar}{2}|e,-\rangle$

Conventionally, quantization axis is along z-axis: $\vec{e} = \hat{e}_z$,
so that

$|\hat{e}_z,+\rangle = |\uparrow\rangle \qquad \hat{S}_z|\uparrow\rangle = +\frac{\hbar}{2}|\uparrow\rangle$

$|\hat{e}_z,-\rangle = |\downarrow\rangle \qquad \hat{S}_s|\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle$

Also useful to define

$$\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y \quad \begin{cases} \hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-) \\ \hat{S}_y = \frac{1}{2i}(\hat{S}_+ - \hat{S}_-) \end{cases}$$

Algebra:

$$[\hat{S}_i, \hat{S}_j] = i\hbar\,\epsilon_{ijk}\,\hat{S}_k$$

$$[\hat{S}_z, \hat{S}_\pm] = \pm\hbar\,\hat{S}_\pm$$

$$[\hat{S}_+, \hat{S}_-] = 2\hbar\,\hat{S}_z$$

Matrix representation

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \hat{S}_+ = \hbar\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

$$\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \qquad \hat{S}_- = \hbar\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

Pauli spinors

$$|\uparrow\rangle \equiv \chi_+ = \begin{pmatrix}1\\0\end{pmatrix}$$

$$|\downarrow\rangle \equiv \chi_- = \begin{pmatrix}0\\1\end{pmatrix}$$

Pauli sigma matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Properties:

$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$ $\qquad \sigma^\dagger = \sigma$ (Hermitian)

$\sigma_x \sigma_y = i\sigma_z$ $\qquad \sigma^\dagger = \sigma^{-1}$ (Unitary)

$\sigma_y \sigma_z = i\sigma_x$ $\qquad \mathrm{Tr}[\sigma] = 0$ traceless

$\sigma_z \sigma_x = i\sigma_y$

If $\vec{A}$ & $\vec{B}$ are two vectors,

$$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot(\vec{A}\times\vec{B})$$

de Moivre formula:

$$U = \exp[i\gamma + i\omega\, \hat{n}\cdot\vec{\sigma}]$$