Wigner Eckart Theorem

Consider rank$-j$ spherical operator $\mathcal{O}_{j,m}$ [of $SU(2)$]
with components $m = \{-j, \ldots, +j\}$

$$[J_z, \mathcal{O}_{j,m}] = m\, \mathcal{O}_{j,m}$$

$$[J_\pm, \mathcal{O}_{j,m}] = \sqrt{(j \mp m)(j \pm m + 1)}\; \mathcal{O}_{j,\, m\pm 1}$$

Theorem:

Clebsch-Gordon

$$\langle \alpha_2; j_2\, m_2 |\, \hat{\mathcal{O}}_{j,m}\, | \alpha_1; j_1, m_1 \rangle = \frac{C^{j_2 m_2}_{j_1 m_1,\, j m}}{\sqrt{2j_2 + 1}} \langle \alpha_2\, j_2 \| \hat{\mathcal{O}}_j \| \alpha_1\, j_1 \rangle$$

or, in terms of Wigner 3j symbol

$$= (-1)^{m_2 + j_1 - j} \begin{pmatrix} j_1 & j & j_2 \\ m_1 & m & -m_2 \end{pmatrix} \langle \alpha_2\, j_2 \| \hat{\mathcal{O}}_j \| \alpha\, j_1 \rangle$$

Wigner 3j symbol:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1 - j_2 - m_3}}{\sqrt{2j_3 + 1}}\; C^{j_3\, -m_3}_{j_1 m_1\; j_2 m_2}$$

inverse relation

$$C^{j_3 m_3}_{j_1 m_1, j_2 m_2} = (-1)^{j_1 - j_2 - m_3} \sqrt{2j_3 + 1} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}$$

See de Shalit & Feshbach (1974)
or Varshalovich (1980s) for more identities