Jenkins – Manohar – Trott minimal coupling trick [arxiv: 1305.0017]

Ingenious (but cheap) way to couple electron-spin to magnetic field, reproducing $g=2$ (!)

Modify $\hat{H} = \frac{p^2}{2m}\mathbb{1} \longrightarrow \frac{(\vec{p}\cdot\vec{\sigma})^2}{2m}$

then apply minimal coupling: $\vec{p} \to \vec{p} - e\vec{A}$ ($\vec{p} \to \vec{p} + e\vec{A}$ for electrons)

$$\hat{H} = \frac{(\vec{p}-e\vec{A})\cdot\vec{\sigma}\;(\vec{p}-e\vec{A})\cdot\vec{\sigma}}{2m}$$

$$= \frac{\vec{p}^2}{2m} + \frac{(\vec{p}\cdot\vec{\sigma})(-e\vec{A})\cdot\vec{\sigma}}{2m} + \frac{(-e\vec{A})\cdot\vec{\sigma}\;(\vec{p}\cdot\vec{\sigma})}{2m} + \frac{(e\vec{A}\cdot\vec{\sigma})^2}{2m}$$

$$= \frac{\vec{p}^2}{2m} - \frac{e}{2m}\left(\underset{\text{commute}}{p_i A_j} + A_i p_j\right)\sigma_i\sigma_j + \frac{e^2A^2}{2m}$$

$$= \frac{\vec{p}^2}{2m} - \frac{e}{2m}\Big(\underbrace{[p_i, A_j]}_{\text{no } i\leftrightarrow j \text{ symmetry}} + \underbrace{A_j p_i + A_i p_j}_{\text{symm. under } i\leftrightarrow j}\Big)\underset{=\frac{1}{2}[\sigma_i,\sigma_j]+\frac{1}{2}\{\sigma_i,\sigma_j\}}{\sigma_i\sigma_j} + \frac{e^2A^2}{2m}$$

$$= \frac{\vec{p}^2}{2m} - \frac{e}{2m}\underbrace{[p_i,A_j]}_{-i\hbar\nabla_i A_j}\frac{1}{2}\underbrace{[\sigma_i,\sigma_j]}_{2i\epsilon_{ijk}\sigma_k} - \frac{e}{2m}\underbrace{[p_i,A_j]}_{-i\hbar\nabla_i A_j}\frac{1}{2}\underbrace{\{\sigma_i,\sigma_j\}}_{2\delta_{ij}} - \frac{e}{2m}2(A_j p_i)\frac{1}{2}\underbrace{\{\sigma_i,\sigma_j\}}_{2\delta_{ij}} + \frac{e^2\vec{A}^2}{2m}$$

$$= \frac{\vec{p}^2}{2m} - \frac{\hbar e}{2m}\underbrace{(\vec{\nabla}\times\vec{A})}_{B}\cdot\underset{\vec{S}=\frac{\hbar\vec{\sigma}}{2}}{\vec{\sigma}} + \frac{i\hbar e}{2m}\nabla\cdot\vec{A} - \frac{e}{m}\vec{A}\cdot\vec{p} + \frac{e^2A^2}{2m}$$

$$= \frac{\vec{p}^2}{2m} - \frac{e}{m}\vec{S}\cdot\vec{B} + \frac{i\hbar e}{2m}\underset{\text{vanishes in Coulomb gauge}}{\nabla\cdot\vec{A}} - \frac{e}{m}\vec{A}\cdot\vec{p} + \frac{e^2\vec{A}^2}{2m}$$

match to $g\frac{-e}{2m}\vec{S}\cdot\vec{B}$

$$\boxed{g=2}$$