Summary of Hydrogen atom

Schrödinger equation:

$$\left[\frac{-\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{x})\right]\psi = E\,\psi$$

(attractive) potential:

$$V(x) = \frac{-1}{4\pi\epsilon_0}\frac{Ze^2}{r} = -\frac{Z\alpha\hbar c}{r}$$

Solution:

$$\psi_{n,l,m}(r,\theta,\phi) = R_{n,l}(r)\,Y_l^m(\theta,\phi)$$

$\boxed{\text{Bound States}}$

$$R_{n,l}(r) = \left(\frac{2}{na}\right)^{2/3}\sqrt{\frac{(n-l-1)!}{2n[(n+l)!]^3}}\,e^{-r/na}\left(\frac{2r}{na}\right)^l L_{n-l-1}^{(2l+1)}\left(\frac{2r}{na}\right)$$

Energy: $E_n = \frac{-mc^2}{2}(Z\alpha)^2\frac{1}{n^2} < 0$ $\qquad n = \{1,2,3,\ldots\}$

$a = \frac{\hbar}{Z\alpha mc}$ atomic Radius $\qquad$ Bohr radius $a_0 = \frac{\hbar}{\alpha mc} = 0.529\,\text{Å}$

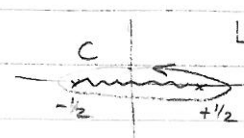
$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}$ Fine structure constant $\qquad 1\text{ Hartree} = \frac{e^2}{4\pi\epsilon_0 a_0} = \alpha^2 mc^2$

$\boxed{\text{Scattering States}}$

$$R_{k,l}(r) = i^l\sqrt{\frac{2}{\pi}}\frac{1}{r}\frac{\Gamma(l+1+i\gamma(k))}{|\Gamma(l+1+i\gamma(k))|}\overbrace{F_l(\gamma(k);\,kr)}^{\text{Regular Coulomb function}}$$

$$= (-i)^l\left(2^{-l-3/2}\pi^{-3/2}\right)\frac{\Gamma(l+1+i\gamma(k))}{e^{\frac{\pi}{2}\gamma(k)}\,k^l}\frac{1}{r^{l+1}}\oint_C\frac{dt\;e^{2ikrt}}{(t-\frac{1}{2})^{l+1-\gamma(k)}\,(t+\frac{1}{2})^{l+1+\gamma(k)}}$$

normalized to satisfy
$\int r^2dr\,R^*R = \delta(k-k')$



Energy: $E_k = \frac{\hbar^2k^2}{2m} > 0$

$\gamma(k) = \frac{mc}{\hbar}\frac{-Z\alpha}{k}$ $\qquad$ Sommerfeld parameter
(negative for attractive potentials)

Note: $\gamma(k) = \frac{-1}{ka}$