

Probability and Current density with $\vec{A} \neq 0$

Start with Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m} \nabla^2 \psi + \frac{i\hbar e}{m} \vec{A} \cdot \nabla \psi + \frac{i\hbar}{2m} (\nabla \cdot \vec{A}) \psi + (\text{other terms}) \psi$$

multiply by ψ^*

$$i\hbar \psi^* \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m} \psi^* \nabla^2 \psi + \frac{i\hbar e}{m} \psi^* \vec{A} \cdot \nabla \psi + \frac{i\hbar}{2m} (\nabla \cdot \vec{A}) \psi^* \psi + (\text{other terms}) \psi^* \psi \quad (*)$$

Construct the same with c.c. of Schrödinger equation

$$-i\hbar \psi \frac{\partial \psi^*}{\partial t} = \frac{-\hbar^2}{2m} \psi \nabla^2 \psi^* - \frac{i\hbar e}{m} \psi \vec{A} \cdot \nabla \psi^* - \frac{i\hbar}{2m} (\nabla \cdot \vec{A}) \psi^* \psi + (\text{other terms}) \psi^* \psi \quad (**)$$

Subtract: $(*) - (**)$

$$i\hbar \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) = \frac{-\hbar^2}{2m} (\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*) + \frac{i\hbar e}{m} (\psi^* \vec{A} \cdot \nabla \psi - \psi \vec{A} \cdot \nabla \psi^*) + \psi^* (\nabla \cdot \vec{A}) \psi - \psi (\nabla \cdot \vec{A}) \psi^*$$

total time derivative
total gradient

$$i\hbar \frac{\partial}{\partial t} (\psi^* \psi) = \frac{-\hbar^2}{2m} \nabla \cdot (\psi^* \nabla \psi - \psi \nabla \psi^*) + \frac{i\hbar e}{m} \nabla \cdot (\psi^* \vec{A} \psi)$$

$$\begin{aligned} \frac{\partial}{\partial t} (\psi^* \psi) &= \frac{i\hbar}{2m} \nabla \cdot (\psi^* \nabla \psi - \psi \nabla \psi^* - \frac{2ie}{\hbar} \psi^* \vec{A} \psi) \\ &= \frac{i\hbar}{2m} \nabla \cdot \left[\psi^* \left(\nabla - \frac{ie}{\hbar} \vec{A} \right) \psi - \psi \left(\nabla + \frac{ie}{\hbar} \vec{A} \right) \psi^* \right] \end{aligned}$$

This is of the form of continuity eqn:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$$

$$\rho = \psi^* \psi$$

$$\vec{J} = \frac{-i\hbar}{2m} \left[\psi^* \left(\nabla - \frac{ie}{\hbar} \vec{A} \right) \psi - \psi \left(\nabla + \frac{ie}{\hbar} \vec{A} \right) \psi^* \right]$$