Properties of Dirac spin matrix $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & \\ & \vec{\sigma} \end{pmatrix} \equiv \sigma^{\mu\nu}$ $\mu, \nu = \{1,2,3\}$.

— Form chosen so as to make Dirac hamiltonian rotationally invariant.
(can obtain it constructively by investigating the representation of Lorentz group)

$$\vec{S}^2 = \left(\frac{\hbar}{2}\right)^2 \vec{\Sigma}^2 = \frac{3}{4}\hbar^2 \mathbb{1}.$$

Algebraic properties:

$$[\alpha_i, \Sigma_j] = \begin{pmatrix} & \sigma_i \\ \sigma_i & \end{pmatrix}\begin{pmatrix} \sigma_j & \\ & \sigma_j \end{pmatrix} - \begin{pmatrix} & \sigma_j \\ \sigma_j & \end{pmatrix}\begin{pmatrix} \sigma_i & \\ & \sigma_i \end{pmatrix}$$

$$= \begin{pmatrix} & \sigma_i\sigma_j \\ \sigma_i\sigma_j & \end{pmatrix} - \begin{pmatrix} & \sigma_j\sigma_i \\ \sigma_j\sigma_i & \end{pmatrix} = \begin{pmatrix} & [\sigma_i, \sigma_j] \\ [\sigma_i, \sigma_j] & \end{pmatrix}$$

$$= 2i\epsilon_{ijk}\begin{pmatrix} & \sigma_k \\ \sigma_k & \end{pmatrix} = 2i\epsilon_{ijk}\alpha_k$$

similarly,

$$\{\alpha_i, \Sigma_j\} = \begin{pmatrix} & \{\sigma_i, \sigma_j\} \\ \{\sigma_i, \sigma_j\} & \end{pmatrix} = 2\delta_{ij}\begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix} = 2\delta_{ij}\,\alpha_0$$

↑ Modern: $\alpha_0 = \gamma_5$

Also:

$$[\alpha_i, \alpha_j] = \begin{pmatrix} & \sigma_i \\ \sigma_i & \end{pmatrix}\begin{pmatrix} & \sigma_j \\ \sigma_j & \end{pmatrix} - \begin{pmatrix} & \sigma_j \\ \sigma_j & \end{pmatrix}\begin{pmatrix} & \sigma_i \\ \sigma_i & \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_i\sigma_j & \\ & \sigma_i\sigma_j \end{pmatrix} - \begin{pmatrix} \sigma_j\sigma_i & \\ & \sigma_j\sigma_i \end{pmatrix} = \begin{pmatrix} [\sigma_i,\sigma_j] & \\ & [\sigma_i,\sigma_j] \end{pmatrix}$$

$$= 2i\epsilon_{ijk}\begin{pmatrix} \sigma_k & \\ & \sigma_k \end{pmatrix} = 2i\epsilon_{ijk}\Sigma_k$$

$$[\beta, \Sigma_i] = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} \sigma_i & \\ & \sigma_i \end{pmatrix} - \begin{pmatrix} \sigma_i & \\ & \sigma_i \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = 0.$$