Kramers – Krönig Relation

- Analyticity $\equiv$ microcausality & unitarity.

Assuming $f(x)$ is analytic in upper-half plane, $x \in \mathbb{R}$, $x_0 \in \mathbb{R}$.
and $f(x) \to 0$ as $|\vec{x}| \to \infty$ then (to be shown)

(*) $$f(x_0) = \frac{1}{i\pi} P \int_{-\infty}^{+\infty} \frac{f(x)}{x - x_0} dx \qquad P \equiv \text{principal value}$$

Split f into real & imaginary parts:

$$\operatorname{Re} f(x_0) + i \operatorname{Im} f(x_0) = \frac{1}{i\pi} P \int_{-\infty}^{+\infty} \frac{\operatorname{Re} f(x_0) + i \operatorname{Im} f(x_0)}{x - x_0}$$

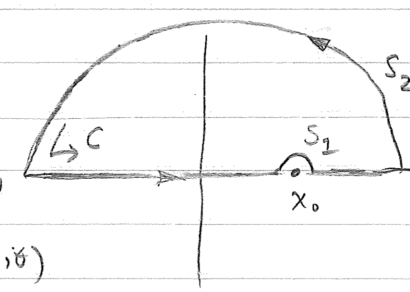
Equate real & imaginary parts: NOTE this i allows relating Real to Imaginary parts (and vice-versa)

$$\left.\begin{aligned} \operatorname{Re} f(x_0) &= \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\operatorname{Im} f(x_0)}{x - x_0} \\ \operatorname{Im} f(x_0) &= \frac{-1}{\pi} P \int_{-\infty}^{+\infty} \frac{\operatorname{Re} f(x_0)}{x - x_0} \end{aligned}\right\} \text{Kramers – Krönig Relations} \quad \text{(Dispersion relations)}$$

To prove (*), close contour in upper half plane since $f(x) \to 0$ there.
Can write:

$$\frac{1}{i\pi} P \int_{-\infty}^{+\infty} \frac{f(x)}{x - x_0} = \frac{1}{i\pi} \Big[\underbrace{\oint_C}_{0 \text{ (analytic)}} - \int_{S_1} - \underbrace{\int_{S_2}}_{0} \Big] \frac{f(x)}{x - x_0}$$

$$= \frac{-1}{i\pi} \int_{S_1} \frac{f(x)}{x - x_0}$$



ch variables: $x - x_0 = re^{i\theta} \Rightarrow dx = i\,d\theta\, re^{i\theta}$

$$= \frac{-1}{\cancel{i}\pi} \int_{\pi}^{0} \cancel{i}\, d\theta\, \cancel{re^{i\theta}} \frac{f(x_0 + re^{i\theta})}{\cancel{re^{i\theta}}}$$

$$= \frac{-1}{\pi} \int_{\pi}^{0} d\theta\, f(x_0 + re^{i\theta})$$

since r is small, expand about $r \approx 0$:

$$= \frac{-1}{\pi} \int_{\pi}^{0} d\theta \left[f(x_0) + O(r) \right]$$

$$= \frac{-1}{\cancel{\pi}} (-\cancel{\pi}) f(x_0) = f(x_0)$$

If the theory is causal, then $G(t-t')$ is zero for negative $t-t'$.
Then,

$$G(t) = \int_{-\infty}^{+\infty} \frac{dk^0}{2\pi} e^{-ik^0 t} \tilde{G}(k^0)$$ [Retarded Green's function]

If $\tilde{G}(k^0)$ is analytic in upper half plane, $G(t<0) = 0$.
Notice for $t<0$,

$$G(t<0) = \int_{-\infty}^{+\infty} \frac{dk^0}{2\pi} e^{+ik^0|t|} \tilde{G}(k^0) = 0.$$

Close in upper-half plane (analytic) $\Rightarrow 0$.

$\Rightarrow$ Retarded Green's functions satisfy kramers-kronig relations.

Flow of Logic: (summary)

Causality $\Rightarrow$ Ret. Green's funct analytic in upper half plane $\Rightarrow$ $G(\omega_0) = \frac{1}{2\pi} P \int_{-\infty}^{\infty} \frac{f(\omega)\,d\omega}{\omega - \omega_0}$ $\Rightarrow$ k-k relations.