Optical Theorem — follows from unitarity of S-matrix

Recall def$^{\underline{n}}$ of S-matrix:

$$\underbrace{\hat{S}}_{} = \underbrace{1}_{\substack{\uparrow \\ \text{Trivial} \\ \text{Scattering}}} + i\underbrace{\hat{T}}_{\substack{\uparrow \\ \text{Transition} \\ \text{matrix}}} \sim 1 + i\underbrace{(2\pi)^4\delta^{(4)}\left(\sum p_{out}^{\mu} - \sum p_{in}^{\mu}\right)}_{\substack{\text{Overall energy-momentum} \\ \text{conserving } \delta\text{-function isolated.}}}\underbrace{\mathcal{M}_{p_{in}\to p_{out}}}_{\substack{\nwarrow \\ \text{Matrix element} \\ \text{(Scattering amplitude)}}}$$

Unitarity of S-matrix, $\hat{S}^\dagger\hat{S} = 1$, implies:

$$(1 + i\hat{T})^\dagger(1 + i\hat{T}) = 1$$

$$1 + i\hat{T} - i\hat{T}^\dagger + \hat{T}^\dagger\hat{T} = 1$$

$$-i(\hat{T} - \hat{T}^\dagger) = \hat{T}^\dagger\hat{T}.$$

Concider $2 \to 2$ particle matrix element

$$-i\left(\langle p_3 p_4|\hat{T}|p_1 p_2\rangle - \langle p_3 p_4|\hat{T}^\dagger|p_1 p_2\rangle\right) = \langle p_3 p_4|T^\dagger T|p_1 p_2\rangle$$

Insert resolution of identity
$\sum_n |n\rangle\langle n| = 1$

$$= \sum_n \underbrace{\frac{1}{n!}}_{\substack{\uparrow \\ \text{Symmetry factor} \\ \text{for indistinguishable} \\ \text{particles}}}\left(\prod_{i=1}^{n}\underbrace{\int\frac{d^4k_i}{(2\pi)^4}}_{\substack{\uparrow \\ \text{Sum over all} \\ \text{possible energy} \\ \text{and momenta...}}}\underbrace{(2\pi)\delta_+(k_i^2 - m_i^2)}_{\substack{\uparrow \\ \text{...subject to} \\ \text{on-shell constraint}}}\right)|\{k_i\}\rangle\langle\{k_i\}|$$

$\delta_+(k_i^2 - m_i^2) \equiv \delta(k_i^2 - m_i^2)\,\Theta(k^0)$

So,

$$-i\left(\langle p_3 p_4|\hat{T}|p_1 p_2\rangle - \overbrace{\langle p_3 p_4|\hat{T}^\dagger|p_1 p_2\rangle}^{\langle p_1 p_2|\hat{T}|p_3 p_4\rangle^*}\right)$$

$$= \sum_n \frac{1}{n!}\left(\prod_{i=1}^{n}\int\frac{d^4k_i}{(2\pi)^4}(2\pi)\delta_+(k_i^2 - m_i^2)\right)\underbrace{\langle p_3 p_4|\hat{T}^\dagger|\{k_i\}\rangle}_{\langle\{k_i\}|\hat{T}|p_3 p_4\rangle^*}\langle\{k_i\}|\hat{T}|p_1 p_2\rangle$$

Separate out the energy-momentum conserving δ-function:

$$-i(2\pi)^4\delta^{(4)}(p_3+p_4-p_1-p_2)\left(\mathcal{M}_{p_1p_2\to p_3p_4}-\mathcal{M}^*_{p_3p_4\to p_1p_2}\right)$$

$$=\sum_n\frac{1}{n!}\left(\prod_{i=1}^{n}\int\frac{d^4k_i}{(2\pi)^4}(2\pi)\delta_+(k_i^2-m_i^2)\right)(2\pi)^4\delta\Big(\sum_i k_i-p_3-p_4\Big)(2\pi)^4\delta\Big(\sum_i k_i-p_1-p_2\Big)$$

$$\times\,\mathcal{M}^*_{p_3p_4\to\{k_i\}}\mathcal{M}_{p_1p_2\to\{k_i\}}$$

Use this δ-function to replace $\sum_i k_i$ with p_3+p_4 in this δ-function

$$-i\,\cancel{(2\pi)^4\delta^{(4)}(p_3+p_4-p_1-p_2)}\left(\mathcal{M}_{p_1p_2\to p_3p_4}-\mathcal{M}^*_{p_3p_4\to p_1p_2}\right)$$

$$=\cancel{(2\pi)^4\delta^{(4)}(p_3+p_4-p_1-p_2)}\sum_n\frac{1}{n!}\left(\prod_{i=1}^{n}\int\frac{d^4k_i}{(2\pi)^4}(2\pi)\delta_+(k_i^2-m_i^2)\right)(2\pi)^4\delta^{(4)}\Big(\sum_i k_i\underbrace{-p_3-p_4}_{-p_1-p_2}\Big)$$

$$\times\,\mathcal{M}^*_{p_3p_4\to\{k_i\}}\mathcal{M}_{p_1p_2\to\{k_i\}}$$

$$-i\left(\mathcal{M}_{p_1p_2\to p_3p_4}-\mathcal{M}^*_{p_3p_4\to p_1p_2}\right)$$

$$=\sum_n\frac{1}{n!}\left(\prod_{i=1}^{n}\int\frac{d^4k_i}{(2\pi)^4}(2\pi)\delta_+(k_i^2-m_i^2)\right)(2\pi)^4\delta^{(4)}\Big(\sum_i k_i-p_1-p_2\Big)\mathcal{M}^*_{p_3p_4\to\{k_i\}}\mathcal{M}_{p_1p_2\to\{k_i\}}$$

Taking (initial state) = (final state) $p_1, p_2 = p_3, p_4$, we have

$$2\,\mathrm{Im}\,\mathcal{M}_{p_1p_2\to p_1p_2}=\sum_n\underbrace{\frac{1}{n!}\left(\prod_{i=1}^{n}\int\frac{d^4k_i}{(2\pi)^4}(2\pi)\delta_+(k_i^2-m_i^2)\right)(2\pi)^4\delta^{(4)}\Big(\sum_i k_i-p_1-p_2\Big)}_{\int d(\mathrm{LIPS})_n}\left|\mathcal{M}_{p_1p_2\to\{k_i\}}\right|^2$$

$$\boxed{2\,\mathrm{Im}\,\mathcal{M}_{p_1p_2\to p_1p_2}=\sum_n\int d(\mathrm{LIPS})_n\left|\mathcal{M}_{p_1p_2\to\{k_1,\dots k_n\}}\right|^2\equiv\underbrace{(\mathrm{Flux})}_{4|\vec{p}_{\mathrm{init}}|\sqrt{s}}\,\sigma_{p_1p_2\to\mathrm{Anything}}}$$

<u>Optical Theorem</u>

Initial & final states identical $\Rightarrow m_1=m_3$ & $m_2=m_4$.

$\Rightarrow$ Forward scattering $\theta_{\mathrm{COM}}=0$ corresponds to $t=0$. $\left(2\,\mathrm{Im}\,\mathcal{M}(s,t=0)=(\mathrm{Flux})\,\sigma_{\mathrm{Total}}\right)$

"(Twice the) imaginary part of the <u>forward</u> (elastic) scattering amplitude is proportional to the (flux times the) <u>total</u> cross section."