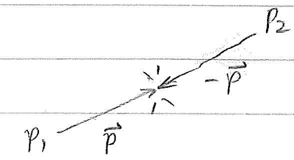
Branch Cuts from unitarity relations

Can carry out the integrations in the $2\to 2$ unitary relation $(m_1 = m_2)$

$$\delta_+(\_) = \delta(\_)\,\theta(E)$$

$$\text{RHS} = \int \frac{d^4k}{(2\pi)^4}\,(2\pi)\,\delta_+(k^2-m^2)\,(2\pi)\,\delta_+\left((p_1+p_2-k)^2-m^2\right)(-1)\,\mathcal{M}^{(+)}\mathcal{M}^{(-)}$$

Move to COM system: $p_1 = (E_1, \vec{p})$ $\quad p_2 = (E_2, -\vec{p})$



then $p_1 + p_2 = (E_1 + E_2;\, 0) = (\sqrt{s},\, 0)$

then
$$(p_1+p_2-k)^2 - m^2 = (p_1+p_2)^2 + k^2 - 2(p_1+p_2)\cdot k - m^2$$
$$= s + \cancel{k^2} - 2\sqrt{s}\,k^0 - \cancel{m^2}$$
$$= s - 2\sqrt{s}\,k^0 \qquad \text{using } \delta_+(k^2 = m^2)$$

$$\text{RHS} = \int \frac{dk^0}{2\pi}\frac{d^3k}{(2\pi)^3}\,(2\pi)\,\delta_+(k^{0\,2}-\vec{k}^2-m^2)\,(2\pi)\,\underbrace{\delta_+(s-2\sqrt{s}\,k^0)}_{\frac{1}{2\sqrt{s}}\delta_+\left(\frac{1}{2}\sqrt{s}-k^0\right)}\,(-1)\,\mathcal{M}^{(+)}\mathcal{M}^{(-)}$$

$$= \frac{1}{2\sqrt{s}}\int \frac{d^3k}{(2\pi)^3}\,(2\pi)\,\delta_+\left(\frac{1}{4}s - \vec{k}^2 - m^2\right)(-1)\,\mathcal{M}\mathcal{M}^*$$

To spherical polars $\quad k^2\,dk \to \frac{1}{2}k\,dk$

$$= \frac{1}{2\sqrt{s}}\int \frac{d\Omega}{(2\pi)^3}\int \frac{1}{2}\,k\,dk^2\,(2\pi)\,\delta_+\left(\frac{1}{4}s - k^2 - m^2\right)(-1)\,\mathcal{M}^{(+)}\mathcal{M}^{(-)}$$

$$= -\frac{\sqrt{\frac{1}{4}s - m^2}}{4(2\pi)^2\sqrt{s}}\int d\Omega\,\mathcal{M}^{(+)}_{p\to 2}\,\mathcal{M}^{(-)}_{2\to p'}$$ (Integrate over all possible angles for the pair of particles with momentum k)

So, in full (multiply by $-i$ on both sides)

$$\boxed{\mathcal{M}^{(+)}_{p\to p'} - \mathcal{M}^{(-)}_{p\to p'} = \frac{i}{32\pi^2}\sqrt{1-\frac{4m^2}{s}}\int d\Omega\,\mathcal{M}^{(+)}_{p\to 2}\,\mathcal{M}^{(-)}_{2\to p'}} \qquad 2\to 2 \text{ unitarity eqn.}$$

$$= \frac{i}{S}\int d(\text{LIPS})_2\,\mathcal{M}^{+}_{p\to 2}\,\mathcal{M}^{-}_{2\to p'}$$