Boundary of Physical Region

The bare minimum enegy required for process $12 \to 34$ to occur is
$E_{min} = m_1 + m_2$ or $m_3 + m_4$ (whichever is greater)

$\Rightarrow$ Physical region: $\begin{cases} s \geq \max\left\{(m_1+m_2)^2,\ (m_3+m_4)^2\right\} \\ \text{AND} \\ -1 \leq \cos\theta \leq 1 \quad \text{(any frame)} \end{cases}$

equivalently, $\cos^2\theta < 1$ or $1-\cos^2\theta > 0$

So, in terms of Mandelstam invariants,
boundary region is given by $1-\cos^2\theta = 0$.

$$1-\cos^2\theta_{com} = 1 - \left[\frac{s^2 + s(2t - \sum m^2) + (m_1^2-m_2^2)(m_3^2-m_4^2)}{\lambda^{1/2}(m_1^2, m_2^2, s)\,\lambda^{1/2}(m_3^2, m_4^2, s)}\right]^2 = 0$$

$$\therefore \frac{1}{4s}\left\{\lambda(m_1^2, m_2^2, s)\lambda(m_3^2, m_4^2, s) - \left[s^2 + s(2t - \sum m^2) + (m_1^2-m_2^2)(m_3^2-m_4^2)\right]^2\right\} = 0$$

$$\therefore -st^2 - s^2t + st\sum m^2 - s(m_1^2-m_3^2)(m_2^2-m_4^2) - t(m_1^2-m_2^2)(m_3^2-m_4^2)$$
$$-(m_1^2 m_4^2 - m_3^2 m_2^2)(m_1^2 - m_2^2 - m_3^2 + m_4^2) = 0 \quad \text{(Mathematica)}$$

Can be compactly written as

$$\phi(s,t) \equiv \frac{1}{2}\,\mathrm{Det}\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & m_1^2 & t & m_2^2 \\ 1 & m_1^2 & 0 & m_3^2 & s \\ 1 & t & m_3^2 & 0 & m_4^2 \\ 1 & m_2^2 & s & m_4^2 & 0 \end{pmatrix} = 0$$

"Gram Determinant"

Generalizes neatly to higher $n \to m$ processes.

Can make LHS more symmetric by replacing $st^2 \to st\left(\sum_i m_i^2 - s - u\right)$

$$\Rightarrow \phi(s,t) = st\,u(s,t) - s(m_1^2-m_3^2)(m_2^2-m_4^2) - t(m_1^2-m_2^2)(m_3^2-m_4^2)$$
$$-(m_1^2 m_4^2 - m_3^2 m_2^2)(m_1^2 - m_2^2 - m_3^2 + m_4^2) = 0.$$