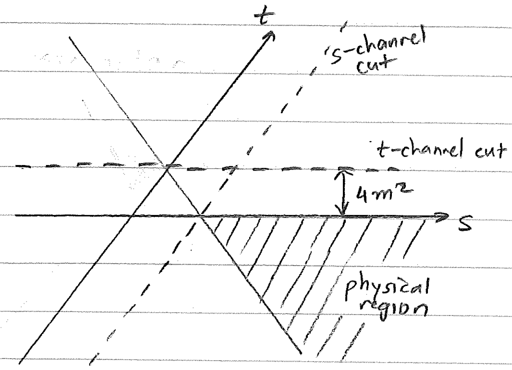
Threshold behavior



Consider partial wave expansion: $(m_1 = m_2 = m_3 = m_4)$

$$A(s,t) = \sum_{\ell=0}^{\infty} (2\ell+1) A_\ell(s)\, P_\ell(z)$$

nearest t singularity to physical region at $t = 4m^2$ (threshold cut)

$\therefore$ expect expansion to converge up to $t_{th} = 4m^2$.

$$\underbrace{\cos\theta}_{z} = \frac{t}{2|p|^2} + 1 \;\Rightarrow\; z_{th} = \frac{4m^2}{2|p|^2} + 1$$

Then, moving left, towards s-channel cut (where $|p| \to 0$), z is growing.

$$\Rightarrow \quad P_\ell(z) \sim z^\ell \sim \left(\frac{4m^2}{2|p|^2} + 1\right)^\ell \sim \left(\frac{4m^2}{2|p|^2}\right)^\ell.$$

This means partial wave amplitude $A_\ell(s)$ must fall with $|p|$ like:

$$A_\ell(s) \sim \left(\frac{2|p|^2}{4m^2}\right)^\ell,$$

but not faster (no higher exponent) since it needs to lose to $P_\ell(z)$ near $t = 4m^2$.

$\therefore$ We obtain:

$*$ $\boxed{A_\ell(s) \sim |p|^{2\ell} \quad \text{as} \quad |p| \to 0}$ (in contrast with N.R. potential scattering $f_\ell \sim |p|^{2\ell+1}$)

Considerations of convergence of partial wave expansion revealed threshold behavior!

Combine this with partial-wave unitarity:

$$\mathrm{Im}\, A_\ell = \frac{|p|}{8\pi\sqrt{s}} |A_\ell|^2 \sim \frac{|p|}{16\pi m}\left(\frac{2|p|^2}{4m^2}\right)^{2\ell} \sim \underline{|p|^{4\ell+1}} \qquad (!)$$

So that using $\mathrm{Im}\, A_\ell = \frac{8\pi}{v}\left[1 - \cos 2\delta_\ell\right]$ $\qquad \cos 2\delta_\ell \sim 1 - 2\delta_\ell^2$

$$|p|^{4\ell+1} = \frac{8\pi m}{|p|}\left[\cancel{1} - (\cancel{1} - 2\delta_\ell^2)\right]$$

$$|p|^{4\ell+2} \sim \delta_\ell^2$$

$$\boxed{\delta_\ell \sim |p|^{2\ell+1}}$$

same as in N.R. potential scattering, as it must!