Lehmann Ellipse

It is not possible to continue the partial wave series in one channel (say s-channel) to the physical region of a crossed channel (say t-channel)

this means, even though $A(s,t)$ is the same in the physical regions of all 3 channels (s,t,u); $A_\ell(s)$ is only valid for physical region of s channel process.

$$A(s,t) = \sum_{\ell=0}^{\infty} (2\ell+1)\, A_\ell(s)\, P_\ell(z(t))$$

Reason: $A(s,t)$ has singularities in both s & t. The singularities in complex s plane appear in partial wave amplitudes $A_\ell(s)$, but since $P_\ell(z)$ are entire functions of z, singularities on the complex t plane appear as divergences of the partial wave series — the series becomes meaningless quickly outside the physical region

Can determine precisely region within which partial wave series converges:

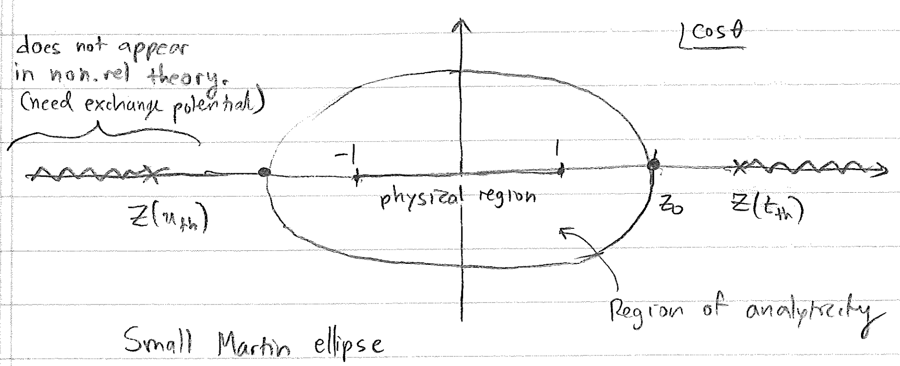
For large ℓ, $\quad \left|P_\ell(z)\right| \xrightarrow[\ell\to\infty]{} \dfrac{f(z)}{\sqrt{\ell}}\, e^{\ell|\mathrm{Im}\,\theta|} \qquad (\ell \in \mathbb{R})$

and we know $\quad A_\ell(s) \xrightarrow[\ell\to\infty]{} \tilde{f}(s)\, e^{-\ell \ln(z_0 + \sqrt{z_0^2-1})} \qquad$ from Froissart–Gribov proj. when deriving Froissart bound.

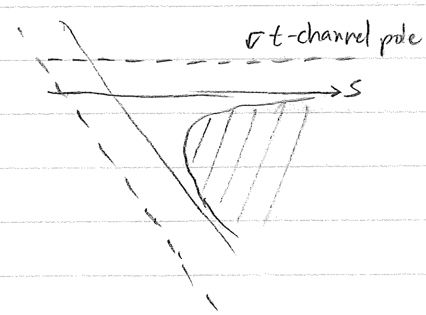
So, series converges if $\quad |\mathrm{Im}\,\theta| \le \ln\left(z_0 + \sqrt{z_0^2-1}\right) \equiv \cosh^{-1}(z_0)$

Follow same steps as in non-relativistic case to find region of analyticity. $\quad \theta \equiv \theta_1 + i\theta_2$

Boundary at $\mathrm{Im}\,\theta = \cosh^{-1}(z_0) \qquad z(\theta_1, \theta_2^{\text{bound}}) = z_0 \cos\theta_1 - i\sqrt{1-z_0^2}\,\sin\theta_1$



Small Martin ellipse



Suppose, instead that ℓ were purely imaginary: $\ell = i|\ell|$.
Then for large ℓ:

$$|P_\ell(z)| \xrightarrow[\ell\to\infty]{} \frac{f(z)}{\sqrt{\ell}}\, e^{|\ell||\mathrm{Re}\,\theta|}.$$

Then the series converges if

$$|\mathrm{Re}\,\theta| \leq \ln\left(z_0 + \sqrt{z_0^2-1}\right) \equiv \cosh^{-1}(z_0)$$

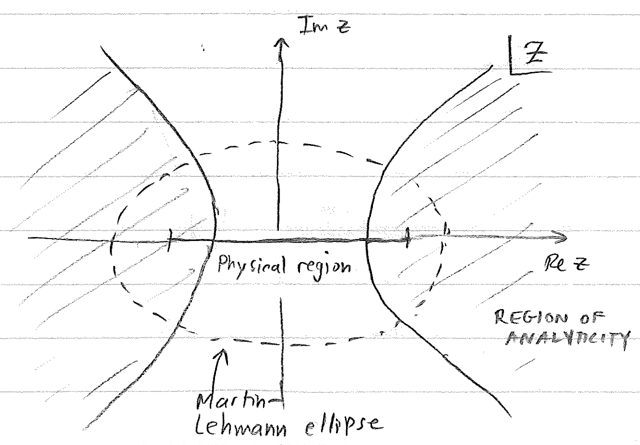
Write $\theta \equiv \theta_1 + i\theta_2$,

then $z(\theta) = \cos\theta = \cos\theta_1 \cosh\theta_2 - i\sin\theta_1 \sinh\theta_2$

at boundary $\cos\theta_1 = \cos\cosh^{-1}(z_0)$
$\sin\theta_1 = \sin\cosh^{-1}(z_0)$

$$z(\theta_1^{\text{bound}}, \theta_2) = \cos\cosh^{-1}(z_0)\cosh\theta_2 - i\sin\cosh^{-1}(z_0)\sinh\theta_2$$

$\rightarrow$ parametric plot: $0 < \theta_2 < 2\pi$ to obtain boundary of analytic region



① Region unbounded.
② Overlaps with original Martin-Lehmann ellipse

① + ② $\Rightarrow$ A continuation to complex ℓ values results in a series that represents the same scattering amplitude.