Weak gauge boson kinetic terms

$f^{abc} = \epsilon^{abc}$ for $SU(2)$.

$$\mathcal{L} = -\frac{1}{4} W^a_{\mu\nu} W^{\mu\nu\, a} \qquad \text{where} \qquad W^a_{\mu\nu} = (\partial_\mu W^a_\nu - \partial_\nu W^a_\mu) - g\epsilon^{abc} W_\mu^b W_\nu^c$$

$$= -\frac{1}{4}\Big[\underbrace{(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)^2}_{1} + \underbrace{g^2 \epsilon^{abc}\epsilon^{ade} W_\mu^b W_\nu^c W^{\mu d} W^{\nu e}}_{2} \underbrace{- 2g\epsilon^{abc}(\partial_\nu W_\mu^a - \partial_\nu W_\mu^a) W^{\mu b} W^{\nu c}}_{3}\Big]$$

Replace: $\boxed{\begin{aligned} W^1_\mu &= \frac{1}{\sqrt{2}}(W^+ + W^-)_\mu \\ W^2_\mu &= \frac{i}{\sqrt{2}}(W^+ - W^-)_\mu \end{aligned}}$ Look at each term seperately.

$$1 = (\partial_\mu W^1_\nu - \partial_\nu W^1_\mu)^2 + (\partial_\mu W^2_\nu - \partial_\nu W^2_\mu)^2 + (\partial_\mu W^3_\nu + \partial_\nu W^3_\mu)^2$$

$$= \frac{1}{2}\Big[\partial_\mu (W^+ + W^-)_\nu - \partial_\nu (W^+ + W^-)_\mu\Big]^2 - \frac{1}{2}\Big[\partial_\mu (W^+ - W^-)_\nu - \partial_\nu (W^+ - W^-)_\mu\Big]^2 + (\partial_\mu W^3_\nu - \partial_\nu W^3_\mu)^2$$

from i^2

$$= \frac{1}{2}(\partial_\mu (W^+ + W^-)_\nu)^2 + \frac{1}{2}(\partial_\nu (W^+ + W^-)_\mu)^2 - \partial_\mu (W^+ + W^-)_\nu \partial^\nu (W^+ + W^-)^\mu$$

$$\underbrace{- \frac{1}{2}(\partial_\mu (W^+ - W^-)_\nu)^2 - \frac{1}{2}(\partial_\nu (W^+ - W^-)_\mu)^2}_{\text{The square terms of these terms cancel in pairs, the cross terms add.}} \underbrace{+ \partial_\mu (W^+ - W^-)_\nu \partial^\nu (W^+ - W^-)^\mu}_{\text{The square terms cancel. cross terms add.}} + (\partial_\mu W^3_\nu - \partial_\nu W^3_\mu)$$

$$= 2\,\partial_\mu W^+_\nu \partial^\mu W^{-\nu} + 2\partial_\nu W^+_\mu \partial^\nu W^{-\nu} - 2\,\partial_\mu W^+_\nu \partial^\nu W^{-\mu} - 2\,\partial_\mu W^-_\nu \partial^\nu W^{+\mu} + (\partial_\mu W^3_\nu - \partial_\nu W^3_\mu)$$

$$= 2\,(\partial_\mu W^+_\nu - \partial_\nu W^+_\mu)(\partial^\mu W^{-\nu} - \partial^\nu W^{-\mu}) + \underbrace{(\partial_\mu W^3_\nu - \partial_\nu W^3_\mu)^2}_{\text{combines with } B_{\mu\nu}B^{\mu\nu} \text{ to give } Z_{\mu\nu}Z^{\mu\nu} \,\&\, F_{\mu\nu}F^{\mu\nu}}$$

kinetic terms for $W^\pm$ & W^3.

$$2 = g^2 \underbrace{\epsilon^{abc}\epsilon^{ade}}_{\delta^{bd}\delta^{ce} - \delta^{be}\delta^{cd}} W_\mu^b W_\nu^c W^{\mu d} W^{\nu e}$$

$$= g^2 (W_\mu^b W^{\mu b} W_\nu^c W^{\nu c} - W_\mu^b W^{\nu b} W_\nu^c W^{\mu c})$$

$$= g^2 \Big((W^1\cdot W^1 + W^2\cdot W^2 + W^3\cdot W^3)^2 - (W^1_\mu W^1_\nu + W^2_\mu W^2_\nu + W^3_\mu W^3_\nu)^2 \Big)$$

used 4-vector dot product to highlight difference in Lorentz index structure.

Now: $W^1_\mu W^1_\nu + W^2_\mu W^2_\nu = \frac{1}{2}(W^+ + W^-)_\mu (W^+ + W^-)_\nu - \frac{1}{2}(W^+ - W^-)_\mu (W^+ - W^-)_\nu$

square terms cancel, cross terms add.

$$= W^+_\mu W^-_\nu + W^-_\mu W^+_\nu$$

So,

$$② = g^2\left((2W^+\cdot W^- + W^3\cdot W^3)^2 - (W^+_\mu W^-_\nu + W^-_\mu W^+_\nu + W^3_\mu W^3_\nu)^2\right)$$

$$= g^2\left(4W^+_\mu W^{-\mu} W^+_\nu W^{-\nu} + 4W^+_\mu W^{-\mu} W^3_\nu W^{3\nu} + W^3_\mu W^{3\mu} W^3_\nu W^{3\nu}\right)$$

cancel

$$- g^2\Big(\underbrace{(W^+_\mu W^-_\nu + W^-_\mu W^+_\nu)^2}_{2W^+_\mu W^-_\nu W^{+\mu} W^{-\nu} + 2W^+_\mu W^-_\nu W^{-\mu} W^{+\nu}} + 2(W^+_\mu W^-_\nu + \underset{\uparrow \text{ rename } \mu\leftrightarrow\nu}{W^-_\mu W^+_\nu})W^{3\mu}W^{3\nu} + W^3_\mu W^3_\nu W^{3\mu} W^{3\nu}\Big)$$

$$= g^2\left(2\left(W^+_\mu W^{-\mu} W^+_\nu W^{-\nu} - W^+_\mu W^-_\nu W^{+\mu} W^{-\nu}\right) + 4\left(W^+_\mu W^{-\mu} W^3_\nu W^{3\nu} - W^+_\mu W^-_\nu W^{3\mu} W^{3\nu}\right)\right)$$

$$= g^2\left(2W^+_\mu W^{-\nu}\left(W^{-\mu} W^+_\nu - W^-_\nu W^{+\mu}\right) + 4W^+_\mu W^{3\nu}\left(W^{-\mu} W^3_\nu - W^-_\nu W^{3\mu}\right)\right)$$

raise/lower ν
factor out -1

raise/lower ν

$$= g^2\left[-(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu)^2 + 2(W^+_\mu W^3_\nu - W^+_\nu W^3_\mu)(W^{-\mu} W^{3\nu} - W^{-\nu} W^{3\mu})\right]$$

Finally,

$$③ = -2g\,\epsilon^{abc}(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)W^{\mu b} W^{\nu c}$$

$$= -4g\left[(\partial_\mu W^1_\nu - \partial_\nu W^1_\mu)W^{2\mu}W^{3\nu} + (\partial_\mu W^2_\nu - \partial_\nu W^2_\mu)W^{3\mu}W^{1\nu} + (\partial_\mu W^3_\nu - \partial_\nu W^3_\mu)W^{1\mu}W^{2\nu}\right]$$

rename $\mu\leftrightarrow\nu$

write in terms of $W^\pm$

$$= -4g\Big[\frac{i}{2}\left(\partial_\mu(W^+ + W^-)_\nu - \partial_\nu(W^+ + W^-)_\mu\right)(W^+ - W^-)^\mu W^{3\nu}$$

$$+ \frac{i}{2}\left(\partial_\nu(W^+ - W^-)_\mu - \partial_\mu(W^+ - W^-)_\nu\right)(W^+ + W^-)^\mu W^{3\nu}$$

terms with W^+W^+ & W^-W^- cancel.
terms with W^+W^- add.

$$+ \frac{i}{2}\underbrace{(\partial_\mu W^3_\nu - \partial_\nu W^3_\mu)}_{\text{antisymmetric in } \mu\leftrightarrow\nu}\underbrace{(W^+ + W^-)^\mu (W^+ - W^-)^\nu}\Big]$$

antisymmetric in $\mu\leftrightarrow\nu$ $\Rightarrow$ only antisymmetric term in $\mu\leftrightarrow\nu$ survives.

$$= -4g\Big[i\big(-\partial_\mu W_\nu^+ W^{-\mu} + \partial_\mu W_\nu^- W^{+\mu} + \partial_\nu W_\mu^+ W^{-\mu} - \partial_\nu W_\mu^- W^{+\mu}\big) W^{3\nu}$$

$$- \frac{i}{2}(\partial_\mu W_\nu^3 - \partial_\nu W_\mu^3)\underbrace{(W^{+\mu}W^{-\nu} - W^{-\mu}W^{+\nu})}_{\text{write as } 2W^{+\mu}W^{-\nu}}\Big]$$

factor out $-i$

$$= 4ig\Big[(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+)W^{-\mu}W^{3\nu} - (W^+ \leftrightarrow W^-) + (\partial_\mu W_\nu^3 - \partial_\nu W_\mu^3)W^{+\mu}W^{-\nu}\Big]$$

So finally, the weak gauge kinetic terms become:

$$\mathcal{L} = -\frac{1}{4} W_{\mu\nu}^a W^{\mu\nu\, a}$$

$$= -\frac{1}{2}(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+)(\partial^\mu W^{-\nu} - \partial^\nu W^{-\mu}) - \frac{1}{4}(\partial_\mu W_\nu^3 - \partial_\nu W_\mu^3)^2$$

$$+ \frac{1}{4} g^2 (W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-)^2 - \frac{1}{2} g^2 (W_\mu^+ W_\nu^3 - W_\nu^+ W_\mu^3)(W^{-\mu}W^{3\nu} - W^{-\nu}W^{3\mu})$$

$$- ig\Big[(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+)W^{-\mu}W^{3\nu} - (W^+ \leftrightarrow W^-) + (\partial_\mu W_\nu^3 - \partial_\nu W_\mu^3)W^{+\mu}W^{-\nu}\Big]$$

For electroweak theory,

write in terms of mass e-states: $W_\mu^3 = \cos\theta_W Z_\mu + \sin\theta_W A_\mu$