Weak charged currents — leptons

Recall: $$l_f = \begin{pmatrix} \begin{pmatrix} \nu_e \\ e \end{pmatrix} \\ \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} \\ \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \end{pmatrix} \qquad \bar{e}_f = \begin{pmatrix} \bar{e} \\ \bar{\mu} \\ \bar{\tau} \end{pmatrix}$$

reference: $H = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$

Since there is no mixing in the lepton sector, consider just one generation:

$$\mathcal{L} = l^\dagger i\bar{\sigma}^\mu D_\mu l + \bar{e}^\dagger i\bar{\sigma}^\mu D_\mu \bar{e} - y_e H.l^\dagger \bar{e}^\dagger + \text{h.c.}$$

write out weak isospin products:

$$\mathcal{L} = (\nu^\dagger \;\; e^\dagger)\, i\bar{\sigma}^\mu \left(\partial_\mu + \frac{ig}{\sqrt{2}} \begin{pmatrix} 0 & W^+ \\ W^- & 0 \end{pmatrix}_\mu + \ldots \right) \begin{pmatrix} \nu \\ e \end{pmatrix} + \bar{e}^\dagger i\bar{\sigma}^\mu D_\mu \bar{e}$$

$$- y_e \,(\phi^+ \;\; \phi^0) . \begin{pmatrix} \nu^\dagger \\ e^\dagger \end{pmatrix} \bar{e}^\dagger + \text{h.c.}$$

$$= \nu^\dagger i\bar{\sigma}^\mu \partial_\mu \nu + e^\dagger i\bar{\sigma}^\mu \partial_\mu e - \frac{g}{\sqrt{2}} W_\mu^+ \nu^\dagger \bar{\sigma}^\mu e - \frac{g}{\sqrt{2}} W_\mu^- e^\dagger \bar{\sigma}^\mu \nu$$

$$- y_e \phi^+ \nu^\dagger \bar{e}^\dagger - y_e \phi^0 e^\dagger \bar{e}^\dagger + \text{h.c.}$$

In four-component Dirac spinor notation,

$$\mathcal{L} = \bar{\nu}(i\partial\!\!\!/)\nu + \bar{e}(i\partial\!\!\!/)e - \frac{g}{\sqrt{2}} W_\mu^+ \bar{\nu}\gamma^\mu \hat{P}_L e - \frac{g}{\sqrt{2}} W_\mu^- \bar{e}\gamma^\mu \hat{P}_L \nu$$

$$= \bar{\nu}(i\partial\!\!\!/)\nu + \bar{e}(i\partial\!\!\!/)e - \frac{g}{2\sqrt{2}} W_\mu^+ \underbrace{\bar{\nu}\gamma^\mu(1-\gamma_5)e}_{V-A} - \frac{g}{2\sqrt{2}} W_\mu^- \bar{e}\gamma^\mu(1-\gamma_5)\nu$$

$$= \bar{\nu}(i\partial\!\!\!/)\nu + \bar{e}(i\partial\!\!\!/)e - \frac{g}{2\sqrt{2}} W_\mu^+ J_\ell^{-\mu} - \frac{g}{2\sqrt{2}} W_\mu^- J_\ell^{+\mu}$$

lepton charged currents

Polar decomposition – Mathematical aside.

Can decompose an arbitrary $n \times n$ complex matrix into a product of a Hermitian matrix $\mathbb{H}$ and a unitary matrix $\mathbb{V}$.
(not nec. special unitary)

$$\mathbb{M} = \mathbb{H}\mathbb{V}$$ (LEFT POLAR DECOMPOSITION)

– See next page for RIGHT POL. DEC.

c.f. polar representation of complex numbers: $z = \rho e^{i\theta}$

matrix	property constraint	independent degrees of freedom
$\mathbb{M}$	—	$\underset{\text{re \& im}}{2}(\underset{\text{row \& column}}{n \times n}) = 2n^2$
$\mathbb{H}$	$\mathbb{H}^\dagger = \mathbb{H}$ (n^2 constraints)	$2n^2 - n^2 = n^2$
$\mathbb{V}$	$\mathbb{V}^\dagger = \mathbb{V}^{-1}$ (n^2 constraints)	$2n^2 - n^2 = n^2$

Unique? Yes, provided $\mathbb{M}$ is invertible ($\det \mathbb{M} \neq 0$)

Construction:

Note $\mathbb{M}\mathbb{M}^\dagger = \mathbb{H}\underbrace{\mathbb{V}\mathbb{V}^\dagger}_{1}\mathbb{H}^\dagger = \mathbb{H}\mathbb{H}^{\dagger}$ ← superfluous. is Hermitian. (& hence, diagonalizable)

Diagonalize with unitary matrix $\mathbb{U}$:

$$\mathbb{U}\mathbb{M}\mathbb{M}^\dagger\mathbb{U}^\dagger = \underset{\text{diagonal matrix}}{\mathbb{D}^2} = \begin{pmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \ddots \end{pmatrix}$$

$\lambda_1^2, \lambda_2^2, \ldots$ positive (or zero)

Moreover, since $\mathbb{M}\mathbb{M}^\dagger = \mathbb{H}\mathbb{H}$ (that is, it's the square of a Hermitian matrix), its eigenvalues are positive semi-definite (i.e. can have zero e-values).

So, $\mathbb{H}$ is the principal square-root of $\mathbb{M}^\dagger\mathbb{M}$: $\mathbb{H} = \sqrt{\mathbb{M}^\dagger\mathbb{M}}$

And therefore the eigenvalues of $\mathbb{H}$ are the positive square-root of those of $\mathbb{M}^\dagger\mathbb{M}$:

$$UHU^{\dagger} = \mathbb{D} = \begin{pmatrix} +\lambda_1 & & \\ & +\lambda_2 & \\ & & \ddots \end{pmatrix} \qquad \lambda_1, \lambda_2, \ldots \text{ positive (or zero).}$$

Hence we can determine $\mathbb{H}$ by inverting the unitary transformation:

$$\mathbb{H} = U^{\dagger} \mathbb{D} U.$$

Having found $\mathbb{H}$, we can get $\mathbb{V}$ by inverting $\mathbb{H}$:

LEFT POLAR DECOMPOSITION or RIGHT-POLAR DECOMPOSITION

$$\mathbb{M} = \mathbb{H}\mathbb{V} \qquad\qquad \mathbb{M} = \mathbb{V}_R \mathbb{H}$$

$$\Rightarrow \boxed{\mathbb{H}^{-1}\mathbb{M} = \mathbb{V}} \qquad\qquad \Rightarrow \boxed{\mathbb{M}\mathbb{H}^{-1} = \mathbb{V}_R}$$

Obviously, this works only if $\mathbb{H}$ (and hence $\mathbb{M}$) is invertible.
If not, the decomposition will not be unique as there will be an ambiguity as to what the phase of the zero eigenvalue is.

c.f. if $z = 0$ then $z = \underset{\substack{\uparrow \\ 0}}{\rho}\, \underbrace{e^{i\theta}}_{?}$ (ambiguous; θ can be anything).

summary of construction: $\mathbb{M} = \mathbb{H}\mathbb{V}$

1. Take product $\mathbb{M}\mathbb{M}^{\dagger}$
2. Diagonalize $U\mathbb{M}\mathbb{M}U^{\dagger} = \mathbb{D}^2$ finding U and $\mathbb{D}^2 = \begin{pmatrix} \lambda_1^2 & \\ & \lambda_2^2 \\ & & \ddots \end{pmatrix}$
3. Take square-root: $\mathbb{D} = +\sqrt{U\mathbb{M}\mathbb{M}^{\dagger}U^{\dagger}}$

determines $\mathbb{H}$ $\rightarrow$ 4. Transform back to get $\mathbb{H} = U^{\dagger}\mathbb{D}U = U^{\dagger}\left(\sqrt{U\mathbb{M}\mathbb{M}^{\dagger}U^{\dagger}}\right)U$

determines $\mathbb{V}$ $\rightarrow$ 5. Invert to get $\mathbb{V} = \mathbb{H}^{-1}\mathbb{V}$ (or $\mathbb{V}_R = \mathbb{M}\mathbb{H}^{-1}$)

Diagonalization of arbitrary $n\times n$ complex matrix $\mathbb{M}$:

Write $\mathbb{M} = \mathbb{H}\mathbb{V}$ (left polar decomposition; $\mathbb{H}$ = Hermitian, $\mathbb{V}$ = Unitary)

Diagonalize $\mathbb{H}$: $\mathbb{H} = U^\dagger D U$

$$\mathbb{M} = U^\dagger \mathbb{D}\, U\mathbb{V}$$ (same U from polar decomposition)

Combination $U\mathbb{V}$ is another unitary matrix U'

$$\mathbb{M} = U^\dagger \mathbb{D}\, U'$$ SINGULAR VALUE DECOMPOSITION OF $\mathbb{M}$

Thus, upon inverting $U^\dagger$ and U'

$$\boxed{U\mathbb{M}U'^\dagger = \mathbb{D} = \begin{pmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \ddots\end{pmatrix}}$$

— $\lambda_1, \lambda_2, \ldots$ are the singular values of $\mathbb{M}$
- can always find U & U' so that $\lambda_1, \ldots$ are real positive

Therefore, in general, it takes two different unitary matrices U & U' to diagonalize an arbitrary $n\times n$ complex matrix:

- One Unitary matrix to render $\mathbb{M}$ hermitian.
- One Unitary matrix to diagonalize it.

a special unitary matrix would suffice. But leave the $U(1)$ phase in, and let physics decide if the phase is needed or not.

n.b the resulting diagonalized matrix has the same eigenvalues as those of $\mathbb{H}$.