Jarlskog Invariant

$V_u M_u V_u^\dagger = D_u \qquad \Rightarrow \qquad M_u = V_u^\dagger D_u V_u \qquad\qquad V_{CKM} = V_u V_d^\dagger$

$V_d M_d V_d^\dagger = D_d \qquad\qquad\quad M_d = V_d^\dagger D_d V_d$

diagonal matrix.

Consider commutator: — measures incompatibility between isospin up and isospin down states

$$
\begin{aligned}
iC &= [M_u, M_d] \\
&= [V_u^\dagger D_u V_u,\ V_d^\dagger D_d V_d] \\
&= V_u^\dagger D_u \underbrace{V_u V_d^\dagger} D_d V_d - V_d^\dagger D_d \underbrace{V_d V_u^\dagger} D_u V_u \\
&= V_u^\dagger \big(D_u V_{CKM} D_d \underbrace{V_d V_u^\dagger}_{V_{CKM}^\dagger} - \underbrace{V_u V_d^\dagger}_{V_{CKM}} D_d V_{CKM}^\dagger D_u\big) V_u \\
&= V_u^\dagger \big(D_u V_{CKM} D_d V_{CKM}^\dagger - V_{CKM} D_d V_{CKM}^\dagger D_u\big) V_u \\
&= V_u^\dagger \big[D_u,\ V_{CKM} D_d V_{CKM}^\dagger\big] V_u
\end{aligned}
$$

Evaluate determinant:

Since C is a commutator, it is Hermitian (after i factored out) and traceless.

The determinant of a (specifically) 3×3 matrix can be written:

$$\det C = \frac{1}{3}\,\mathrm{Tr}[C^3] \qquad \left[\text{Special case of Cayley-Hamilton theorem}\right]$$

$$
\begin{aligned}
&= \frac{1}{3}(-i)^3\,\mathrm{Tr}\big([M_u, M_d]^3\big) \\
&= \frac{i}{3}\,\mathrm{Tr}\Big((M_u M_d - M_d M_u)^2 (M_u M_d - M_d M_u)\Big) \\
&= \frac{i}{3}\,\mathrm{Tr}\Big(\big(M_u M_d M_u M_d - M_u M_d M_d M_u - M_d M_u M_u M_d + M_d M_u M_d M_u\big) \\
&\qquad\qquad \times (M_u M_d - M_d M_u)\Big)
\end{aligned}
$$

short hand: $u \equiv M_u$, $d \equiv M_d$

$$
\begin{aligned}
&= \frac{i}{3}\,\mathrm{Tr}\Big(\cancel{ududud} - udduud - duudud + duduud \\
&\qquad\qquad - ududdu + uddudu + duuddu - \cancel{dududu}\Big)
\end{aligned}
$$

use cyclic property of trace to simplify.

$$\det \mathbb{C} = \frac{i}{3} \; 3 \, \mathrm{Tr}\left[M_u^2 M_d^2 M_u M_d - M_d^2 M_u^2 M_d M_u \right]$$

$$= \mathrm{Tr}\left[i M_u^2 M_d^2 M_u M_d - i M_d^2 M_u^2 M_d M_u \right]$$

Notice 2nd term is $(\dagger)$ of 1st term, after cyclic permutation.
(given that $M_u^\dagger = M_u$ & $M_d^\dagger = M_d$)

$M_u M_d M_u^2 M_d^2$ ✓

$$= 2i \, \mathrm{Im}\left[i M_u^2 M_d^2 M_u M_d \right]$$

$a - a^* = 2i \, \mathrm{Im}(a)$

Insert definition in terms of diagonalized mass matrices.

cyclic.

$$= -2 \, \mathrm{Im}\left[(V_u^\dagger D_u^2 V_u)(V_d^\dagger D_d^2 V_d)(V_u^\dagger D_u V_u)(V_d^\dagger D_d V_d) \right]$$

$$= -2 \, \mathrm{Im}\left[D_u^2 V_{CKM} D_d^2 V_{CKM}^\dagger D_u V_{CKM} D_d V_{CKM}^\dagger \right]$$

$$= -2 \, \mathrm{Im} \sum_{i,j} \sum_{\alpha,\beta} (m_u^2)_i \, V_{i\alpha} \, (m_d^2)_\alpha \, V_{j\alpha}^* \, (m_u)_j \, V_{j\beta} \, (m_d)_\beta \, V_{i\beta}^*$$

indices $i, j, \ldots$ = up flavor
indices $\alpha, \beta, \ldots$ = down flavor

$$= -2 \sum_{i,j} \sum_{\alpha,\beta} \underbrace{\mathrm{Im}\, V_{i\alpha} V_{j\alpha}^* V_{j\beta} V_{i\beta}^*}_{\parallel} \; \underbrace{(m_u^2)_i (m_d^2)_\alpha (m_u)_j (m_d)_\beta}_{\text{real-valued}}$$

$$J \sum_{\gamma,k} \epsilon_{\alpha\beta\gamma} \epsilon_{ijk}$$

$J \equiv$ Jarlskog invariant

$$= -2 J \underbrace{\sum_{i,j,k} (m_u^2)_i (m_u)_j}_{} \sum_{\alpha\beta\gamma} (m_d^2)_\alpha (m_d)_\beta$$

$$= -2 J \left[(m_t - m_c)(m_t - m_u)(m_c - m_u) \right] \left[(m_b - m_s)(m_b - m_d)(m_s - m_d) \right]$$

$$= -2 J \underbrace{F_u \, F_d}_{\text{Jarlskog det.}}$$

Jarlskog invariant