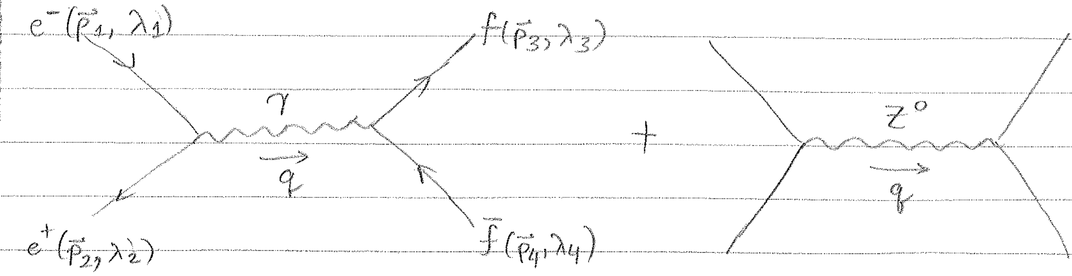
$e^+e^- \longrightarrow f\bar{f}$ [γ–Z interference]

Consider any fermion f except electrons (which has t-channel diagrams)



– At $\sqrt{s} \sim M_Z$, neglect fermion masses.

$\Rightarrow$ Work with helicity states ($\equiv$ chirality)

Angular momentum (or helicity) conservation requires particle/anti-particle to be spin-triplet state $\Rightarrow$ $(\lambda_1 = -\lambda_2 \equiv \lambda_e \;;\; \lambda_3 = -\lambda_4 \equiv \lambda_f)$

(same chirality)

Compute amplitude matrix: $M = \begin{pmatrix} M_{L\leftarrow L} & M_{L\leftarrow R} \\ M_{R\leftarrow L} & M_{R\leftarrow R} \end{pmatrix}$

vanish by W.I.

$$iM_{\lambda_f \leftarrow \lambda_e} = \bar{v}(\vec{p}_2)(-ieQ_e\gamma^\mu)\hat{P}_{\lambda_e}u(\vec{p}_1)\,\frac{-i}{q^2}\left(g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}\right)\bar{u}(\vec{p}_3)(-ieQ_e\gamma^\nu)\hat{P}_{\lambda_f}v(\vec{p}_4)$$

$$+\,\bar{v}(\vec{p}_2)\left[-i g_Z\gamma^\mu g_\lambda^{(e)} P_{\lambda_e}\right]u(\vec{p}_1)\,\frac{-i}{q^2 - M_Z^2 - iM_Z\Gamma}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{M_Z^2}\right)\bar{u}(\vec{p}_3)\left(-ig_Z\gamma^\nu g_\lambda^{(f)}\right)\hat{P}_{\lambda_f}v(\vec{p}_4)$$

$$= i\,\frac{(eQ_e)(eQ_f)}{q^2}\;\bar{v}(\vec{p}_2)\gamma^\mu\hat{P}_{\lambda_e}u(\vec{p}_1)\;\bar{u}(\vec{p}_3)\gamma_\mu\hat{P}_{\lambda_f}v(\vec{p}_4)$$

$$+\,i\,\frac{(g_Z g_\lambda^{(e)})(g_Z g_\lambda^{(f)})}{q^2 - M_Z^2 - iM_Z\Gamma}\;\bar{v}(\vec{p}_2)\gamma^\mu\hat{P}_{\lambda_e}u(\vec{p}_1)\;\bar{u}(\vec{p}_3)\gamma_\mu\hat{P}_{\lambda_f}v(\vec{p}_4)$$

Virtue of working in chiral basis — amplitude neatly factorizes.

$$iM_{\lambda_f \leftarrow \lambda_e} = i\,G_{\lambda_f \leftarrow \lambda_e}(q^2)\left(\bar{v}_{p_2}\gamma^\mu\hat{P}_{\lambda_e}u_{p_1}\right)\left(\bar{u}_{p_3}\gamma_\mu\hat{P}_{\lambda_f}v_{p_4}\right)$$

where $G_{\lambda_f \leftarrow \lambda_e}(s) = \sum\limits_{V=\gamma, Z}\dfrac{g_\lambda^{(f)}(Z\to f\bar{f})\;g_\lambda^{(e)}(Z\to e^-e^+)}{s - M_V^2 + iM_V\Gamma_V}$

Compute polarized diff. cross section.

Square amplitude: safe to sum over final spins (NOT average.) } because of inserted projectors
" " sum over final spins
sum over final state colors

$$|\mathcal{M}|^2_{\lambda_f\leftarrow\lambda_e} = \left|G_{\lambda_f\leftarrow\lambda_e}(q^2)\right|^2 \sum_{s_1 s_2}(\bar v_{p_2}\gamma^\mu \hat P_{\lambda_e} u_{p_1})(\bar u_{p_1}\gamma^\nu \hat P_{\lambda_e} v_{p_2})$$

$$\times N_c \sum_{s_3 s_4}(\bar u_{p_3}\gamma_\mu \hat P_{\lambda_f} v_{p_4})(\bar v_{p_4}\gamma_\nu \hat P_{\lambda_f} u_{p_3})$$

$$= \left|G_{\lambda_f\leftarrow\lambda_e}(q^2)\right|^2 \;\mathrm{Tr}\left[\not p_2 \gamma^\mu \hat P_{\lambda_e} \not p_1 \gamma^\nu \hat P_{\lambda_e}\right] N_c\, \mathrm{Tr}\left[\not p_3 \gamma_\mu \hat P_{\lambda_f} \not p_4 \gamma_\nu \hat P_{\lambda_f}\right]$$

$$= \left|G_{\lambda_f\leftarrow\lambda_e}(q^2)\right|^2 \left(2p_1^\nu p_2^\mu + 2p_1^\mu p_2^\nu - 2p_1\cdot p_2\, g^{\mu\nu} + 4i\lambda_e\, p_{1\alpha}p_{2\beta}\,\epsilon^{\mu\nu\alpha\beta}\right)$$

$$\times N_c\left(2p_{3\nu}p_{4\mu} + 2p_{3\mu}p_{4\nu} - 2p_3\cdot p_4\, g_{\mu\nu} + 4i\lambda_f\, p_3^\gamma p_4^\delta\,\epsilon_{\mu\nu\gamma\delta}\right)$$

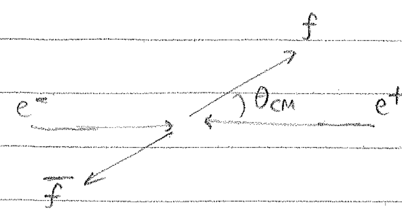
Contract, and apply Mandelstam relations (eliminate s in favor of t & u)

$$= N_c \left|G_{\lambda_f\leftarrow\lambda_e}(q^2)\right|^2\, 2\left[t^2+u^2 - 4\lambda_e\lambda_f\,(t^2-u^2)\right] \qquad \boxed{\lambda_e, \lambda_f = \{\pm 1/2\}}$$

$$= N_c \left|G_{\lambda_f\leftarrow\lambda_e}(q^2)\right|^2 \begin{cases} 4t^2, & \lambda_f, \lambda_e \text{ same sign} \\ 4u^2, & \lambda_f, \lambda_e \text{ opposite sign.} \end{cases}$$

kinematics

$$t = \frac{-s}{2}(1-\cos\theta_{cm});$$
$$u = \frac{-s}{2}(1+\cos\theta_{cm})$$
$$q^2 = s$$



!! Note how scattering angle is defined

$$= N_c \left|G_{\lambda_f\leftarrow\lambda_e}(s)\right|^2 s^2\left[1+\cos^2\theta - 8\lambda_e\lambda_f\cos\theta\right]$$

Differential cross section:

$$d\sigma_{\lambda_f\leftarrow\lambda_i} = \frac{1}{2s}\,|\mathcal{M}|^2_{\lambda_f\leftarrow\lambda_e}\,\frac{1}{32\pi^2}\,d\Omega$$

$$\frac{d\sigma_{\lambda_f \leftarrow \lambda_e}}{d\cos\theta} = \frac{1}{2\cancel{s}} \quad s^{\cancel{2}} \frac{1}{32\pi^2} \cdot 2\pi \quad N_c \left| G_{\lambda_f \leftarrow \lambda_e}(s) \right|^2 \left[1 + \cos^2\theta + 8\lambda_e \lambda_f \cos\theta \right]$$

$$= \frac{s}{32\pi} N_c \left| G_{\lambda_f \leftarrow \lambda_e}(s) \right|^2 \left[1 + \cos^2\theta + 8\lambda_e \lambda_f \cos\theta \right]$$

When integrated over $\theta : 0 \to \pi$,

$\lambda_e \lambda_f \cos\theta$ term vanishes (odd under $\cos\theta \to -\cos\theta$)

$$\boxed{\begin{aligned} \sigma_{\lambda_f \leftarrow \lambda_i} &= \frac{s}{32\pi} N_c \left| G_{\lambda_f \leftarrow \lambda_e}(s) \right|^2 \frac{8}{3} \\ &= \frac{N_c}{12\pi} s \left| G_{\lambda_f \leftarrow \lambda_e}(s) \right|^2 \end{aligned}}$$

TOTAL POLARIZED CROSS SECTION FOR GIVEN FINAL-STATE FERMION f

λ_e labels helicity of initial state e^- and helicity of initial state e^+

λ_f labels helicity of final state f and helicity of final state $\bar{f}$

($= \pm 1/2$)

$$\left| G_{\lambda_f \leftarrow \lambda_e}(s) \right|^2 = \frac{e^4 Q_f^2 Q_e^2}{s^2} + \frac{g^4}{\cos^4\theta_W} \frac{(g_\lambda^{(f)})^2 (g_\lambda^{(e)})^2}{(s - M_Z^2)^2 + M_Z^2 \Gamma^2}$$

$$+ 2 \frac{e^2 Q_f Q_e}{s} \frac{g^2}{\cos^2\theta_W} \frac{g_\lambda^{(f)} g_\lambda^{(e)}}{(s - M_Z^2)^2 + M_Z^2 \Gamma^2} \underbrace{(s - M_Z^2)}_{\text{vanishes at } Z\text{-pole.}}$$

Approximation:

At Z-pole ($\sqrt{s} = M_Z$)

$$\left| G_{\lambda_f \leftarrow \lambda_e}(M_Z^2) \right|^2 = \underbrace{\frac{e^4 Q_f^2 Q_e^2}{M_Z^4}}_{\text{negligible } O\left(\frac{\Gamma_Z^2}{M_Z^2}\right)} + \frac{g^4}{\cos^4\theta_W} \frac{(g_\lambda^{(f)})^2 (g_\lambda^{(e)})^2}{M_Z^2 \Gamma^2}$$

$$\simeq \frac{g^4}{\cos^4\theta_W} \frac{g_\lambda^{(f)2} g_\lambda^{(e)2}}{M_Z^2 \Gamma^2}$$

with $g_\lambda^{(f)} = \underbrace{T_f^3}_{\substack{\text{vanishes if } \lambda = +1/2 \\ \text{(right-handed)}}} - Q_f \sin^2\theta_W$

Pole function

$$G_{\lambda_f \leftarrow \lambda_i}(s) = \frac{e^2 Q_i Q_f}{s} + \frac{g^2}{\cos^2\theta_W} \frac{g^f_\lambda \, g^i_\lambda}{s - M_Z^2 + i M_Z \Gamma}$$

rationalize:

$$= \frac{e^2 Q_i Q_f}{s} + \frac{g^2}{\cos^2\theta_W} \frac{g^f_\lambda \, g^i_\lambda}{(s - M_Z^2)^2 + M_Z^2 \Gamma^2} \left(s - M_Z^2 - i M_Z \Gamma\right)$$

Square-magnitude:

$$\left| G_{\lambda_f \leftarrow \lambda_i}(s) \right|^2 = \frac{e^4 Q_i Q_f}{s^2} + \frac{e^2 Q_i Q_f}{s} \frac{g^2}{\cos^2\theta_W} \frac{g^f_\lambda \, g^i_\lambda}{(s - M_Z^2)^2 + M_Z^2 \Gamma^2} 2(s - M_Z^2)$$

$$+ \frac{g^4}{\cos^4\theta_W} \frac{(g^f_\lambda g^i_\lambda)^2}{(s - M_Z^2)^2 + M_Z^2 \Gamma^2}$$

Approximations:

① Well below Z pole $\boxed{\sqrt{s} \ll M_Z}$ — Neglect last term.

$$\left| G_{\lambda_f \leftarrow \lambda_i}(s) \right|^2 \approx \frac{e^4 Q_f^2 Q_i^2}{s^2} \left[1 - \frac{2}{e^2 Q_f^2 Q_i^2} \frac{g^2 \, g^f_\lambda g^i_\lambda}{\cos^2\theta_W} \left(\frac{s}{M_Z^2} \right) + \dots \right]$$

② At Z pole $\boxed{\sqrt{s} = M_Z}$ — neglect γ contribution. (first two terms)

$$\left| G_{\lambda_f \leftarrow \lambda_i}(s) \right|^2 \approx \frac{g^4}{\cos^4\theta_W} \frac{(g^f_\lambda g^i_\lambda)^2}{M_Z^2 \Gamma^2} + \mathcal{O}\left(\frac{\Gamma^2}{M_Z^2} \right)$$

③ Well above Z pole $\boxed{\sqrt{s} \gg M_Z}$

— all three terms equal in magnitude.