Lie Algebra

Definition:

Consists of vector space L over a field ($\mathbb{R}$ or $\mathbb{C}$, eg) with a composition rule, called product, denoted $\circ$, defined

$$\circ: \quad L \times L \longrightarrow L$$

If $a_1, a_2, a_3 \in L$ are any three elements in L, then the following defines the Lie algebra:

1. Closure $\quad a_1 \circ a_2 \in L$
2. Linear $\quad a_1 \circ (a_2 + a_3) = a_1 \circ a_2 + a_1 \circ a_3$
3. Antisymmetry $\quad a_1 \circ a_2 = -a_2 \circ a_1$
4. Jacobi identity:

$$a_1 \circ (a_2 \circ a_3) + a_3 \circ (a_1 \circ a_2) + a_2 \circ (a_3 \circ a_1) = 0$$

For example, $\{\sigma_1, \sigma_2, \sigma_3\}$ (Pauli sigma matrices) satisfies Lie Algebra properties, where $\circ$ is commutator.

Graded Algebra

$\mathbb{Z}_2$ Graded:

Consists of vector space L which is the direct sum of two subspaces, L_0, L_1:

$$L = L_0 \oplus L_1$$

and a product

$$\circ : L \times L \rightarrow L$$

with the following propertres

Let $a_1, a_2 \in L_0$ and $b_1, b_2 \in L_1$

$$\left.\begin{array}{l} ① \ a_1 \circ a_2 \in L_0 \\ ② \ a_1 \circ b_1 \in L_1 \\ ③ \ b_1 \circ b_2 \in L_0 \end{array}\right\} \text{sense of even \& odd: } a \text{ is even, } b \text{ is odd.}$$

Example:

The set of all integers, $\mathbb{Z}$, under addition.

Here, (even integers) $\in L_0$ even + even = even
(odd integers) $\in L_1$ ~~even + odd = odd~~
odd + odd = even ✓

$\mathbb{Z}_n$ Graded:

– direct sum of $N+1$ vector spaces: $L = L_0 \oplus L_1 \oplus L_2 \dots \oplus L_N$

with product: $\circ : L \times L \longrightarrow L$.

With property:

Let $a_k \in L_k$. then $\underbrace{a_j \circ a_k \in L_{j+k, \bmod (N+1)}}_{\text{with this property, it's called grading.}}$