Coherent states for Fermionic oscillators

[Works only for Dirac fermions]

$$\{\hat{\psi}, \hat{\psi}^\dagger\} = \hbar \qquad \{\hat{\psi}, \hat{\psi}\} = \{\hat{\psi}^\dagger, \hat{\psi}^\dagger\} = 0.$$

Convenient to normalize $\boxed{\hat{\psi} = \sqrt{\hbar}\,\hat{c},}$ so that:

(so $\hat{c}$ & $\hat{c}^\dagger$ are unitless)

$$\{\hat{c}, \hat{c}^\dagger\} = 1 \quad \text{and} \quad \{\hat{c}, \hat{c}\} = \{\hat{c}^\dagger, \hat{c}^\dagger\} = 0.$$

— can be represented on a 2-dimensional Hilbert space $= \text{span}\{|0\rangle, |1\rangle\}$:

$$\hat{c} = \begin{pmatrix} \langle 0|c|0\rangle & \langle 0|c|1\rangle \\ \langle 1|c|0\rangle & \langle 1|c|1\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{"annihilation operator"} \quad c|0\rangle = 0.$$

$$\hat{c}^\dagger = \begin{pmatrix} \langle 0|c^\dagger|0\rangle & \langle 0|c^\dagger|1\rangle \\ \langle 1|c^\dagger|0\rangle & \langle 1|c^\dagger|1\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{"creation operator"} \quad c^\dagger|0\rangle = |1\rangle$$

Define coherent state:

Phase is a Grassmann odd number

(anticommutes with $\hat{c}$ & $\hat{c}^\dagger$)

$$|\xi\rangle = e^{\hat{c}^\dagger \xi}|0\rangle = (1 + \hat{c}^\dagger \xi)|0\rangle = |0\rangle - \xi|1\rangle = |0\rangle + |1\rangle\xi.$$

c.c.

$$\langle \xi| = \langle 0| e^{\xi^* \hat{c}} = \langle 0|(1 + \xi^* \hat{c}) = \langle 0| - \langle 1|\xi^* = \langle 0| + \xi^*\langle 1|$$

Properties:

① Eigenstate of annihilation operator: (fixes the order in exponent $e^{\hat{c}^\dagger \xi}$)

$$\hat{c}|\xi\rangle = \xi|\xi\rangle$$

proof:
$$\begin{aligned} &= \hat{c}\left(|0\rangle + |1\rangle\xi\right) \quad \text{(Final form of } |\xi\rangle\text{)} \\ &= 0 - |0\rangle\xi \\ &= \xi|0\rangle \\ &= \xi\left(|0\rangle - |1\rangle\xi\right) \quad (\text{because } \xi^2 = 0) \\ &= \xi|\xi\rangle. \end{aligned}$$

② Overlap

$$\langle \zeta | \xi \rangle = e^{\zeta^* \xi}$$

proof:

$$= \big(\langle 0| + \zeta^* \langle 1|\big)\big(|0\rangle + |1\rangle \xi\big)$$

$$= (1 + \zeta^* \xi)$$

$$= e^{\zeta^* \xi} \checkmark$$

Not orthogonal

③ Resolution of identity

$$\hat{1} = \int d\xi^* d\xi \; e^{-\xi^* \xi} \, |\xi\rangle\langle \xi|$$

proof:

$$\int d\xi^* d\xi \, (1 - \xi^*\xi)\big(|0\rangle + |1\rangle \xi\big)\big(\langle 0| + \xi^* \langle 1|\big)$$

$$= \int d\xi^* d\xi \, (1 - \xi^*\xi)\big(|0\rangle\langle 0| + \xi^* |0\rangle\langle 1| + |1\rangle\langle 0| \xi + |1\rangle \xi \xi^* \langle 1|\big)$$

$$= \int d\xi^* d\xi \, \Big(\underbrace{|0\rangle\langle 0| + \xi^* |0\rangle\langle 1| + |1\rangle\langle 0|\xi}_{\text{integrates to zero.}} - \xi^*\xi \, |1\rangle\langle 1| - \xi^*\xi \, |0\rangle\langle 0|\Big)$$

$$= \underbrace{\int d\xi^* d\xi \, (-\xi^*\xi)}_{=1} \big(|0\rangle\langle 0| + |1\rangle\langle 1|\big) \checkmark$$

④ Trace:

$$\operatorname{Tr} \hat{A} = \int d\xi^* d\xi \; e^{-\xi^*\xi} \langle -\xi | \hat{A} | \xi \rangle$$

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proof:

$$= \int d\xi^* d\xi \, (1 - \xi^*\xi)\big(\langle 0| - \xi^* \langle 1|\big) \hat{A} \big(|0\rangle + |1\rangle \xi\big)$$

$$= \int d\xi^* d\xi \, (1 - \xi^*\xi)\big(\langle 0|\hat{A}|0\rangle + \langle 0|\hat{A}|1\rangle \xi - \xi^* \langle 1|\hat{A}|0\rangle - \xi^* \langle 1|\hat{A}|1\rangle \xi\big)$$

$$= \int d\xi^* d\xi \, \Big(\underbrace{\cdots}_{\text{integrates to zero.}} - \xi^*\xi \langle 1|\hat{A}|1\rangle - \xi^*\xi \langle 0|\hat{A}|0\rangle\Big)$$

$$= \underbrace{\int d\xi^* d\xi \, (-\xi^*\xi)}_{1} \big(\langle 0|\hat{A}|0\rangle + \langle 1|\hat{A}|1\rangle\big) \checkmark$$

(5) Supertrace:

Recall definition: $\mathrm{STr}(\hat{A}) = \mathrm{Tr}\left[(-1)^{\hat{F}}\,\hat{A}\right]$ $\qquad \hat{F}$ = fermion parity.

$$\mathrm{STr}\,\hat{A} = \int d\xi^* d\xi\, e^{-\xi^*\xi} \langle +\xi|\hat{A}|\xi\rangle$$

proof: identical to (4), with change $- \to +$

⋮

$$= \underbrace{\int d\xi^* d\xi \left(-\xi^*\xi\right)}_{+1} \left(\langle 0|\hat{A}|0\rangle - \langle 1|\hat{A}|1\rangle\right) \quad ✓$$