Path integral representation of Dirac-fermionic oscillator

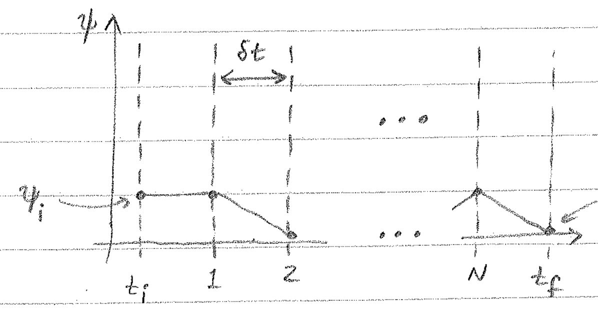
Goal: Obtain quantum mechanical kernel

$$A_{\psi_f \leftarrow \psi_i} = \langle \psi_f | e^{-\frac{i}{\hbar}\hat{H}(t_f - t_i)} | \psi_i \rangle$$

Assume Hamiltonian is normal ordered — $\hat{\psi}$ to the left of $\hat{\psi}^\dagger$,

e.g.: $\hat{H} = \omega \hat{\psi}^\dagger \hat{\psi} + \ldots$

Split $t_f - t_i$ into N intervals, duration δt. $\quad t_f - t_i = \delta t (N+1)$



each factor corresponds to a line segment.

then

$$A_{\psi_f \leftarrow \psi_i} = \langle \psi_f | e^{-\frac{i}{\hbar}\hat{H}\delta t} e^{-\frac{i}{\hbar}\hat{H}\delta t} \cdots \underbrace{e^{-\frac{i}{\hbar}\hat{H}\delta t}} | \psi_i \rangle$$

each resolution corresponds to a node.

Insert resolution of identity between each pair of $e^{-\frac{i}{\hbar}\hat{H}\delta t}$'s

$$\mathbb{1} = \int \frac{d\psi_1^* d\psi_1}{\hbar} e^{-\frac{1}{\hbar}\psi_1^* \psi_1} |\psi_1\rangle\langle\psi_1| \quad \text{between last pair}$$

$$\mathbb{1} = \int \frac{d\psi_2^* d\psi_2}{\hbar} e^{-\frac{1}{\hbar}\psi_2^* \psi_2} |\psi_2\rangle\langle\psi_2| \quad \text{between penultimate pair}$$

$$\vdots$$

$$\mathbb{1} = \int \frac{d\psi_N^* d\psi_N}{\hbar} e^{-\frac{1}{\hbar}\psi_N^* \psi_N} |\psi_N\rangle\langle\psi_N| \quad \text{between first pair}$$

so that

$$A_{\psi_f \leftarrow \psi_i} = \langle \psi_f | e^{-\frac{i}{\hbar}\hat{H}\delta t} \int \frac{d\psi_N^* d\psi_N}{\hbar} e^{-\psi_N^* \psi_N} |\psi_N\rangle\langle\psi_N| e^{-\frac{i}{\hbar}\hat{H}\delta t} \cdots$$

$$\cdots \int \frac{d\psi_1^* d\psi_1}{\hbar} e^{-\frac{1}{\hbar}\psi_1^* \psi_1} |\psi_1\rangle\langle\psi_1| e^{-\frac{i}{\hbar}\hat{H}\delta t} |\psi_i\rangle$$

Collect integrals (Grassmann even measure); bring to front.

$$A_{\psi_f \leftarrow \psi_i} = \int \left(\prod_{k=1}^{N} \frac{d\psi_k^* d\psi_k}{\hbar} e^{-\frac{1}{\hbar}\psi_k^*\psi_k} \right) \langle \psi_f | e^{-\frac{i}{\hbar}\hat{H}\delta t} \cdots |\psi_k\rangle \langle \psi_k | e^{-\frac{i}{\hbar}\hat{H}\delta t} | \psi_{k-1}\rangle \langle \psi_{k-1}| \cdots e^{-\frac{i}{\hbar}\hat{H}\delta t} |\psi_i\rangle$$

$$= \int \left(\prod_{k=1}^{N} \frac{d\psi_k^* d\psi_k}{\hbar} e^{-\frac{1}{\hbar}\psi_k^* d\psi_k} \right) \left(\prod_{k=1}^{N+1} \langle \psi_k | e^{-\frac{i}{\hbar}\hat{H}\delta t} |\psi_{k-1}\rangle \right)$$

where: $|\psi_{k=0}\rangle = |\psi_i\rangle$, $|\psi_{k=N+1}\rangle = |\psi_f\rangle$

Amplitude is a string of $(N+1)$ bra-ket factors.

Consider a typical bra-ket. Use fact that $\hat{H}$ is normal ordered:

$$\langle \psi_k | e^{-\frac{i}{\hbar}H(\hat{\psi}^\dagger, \hat{\psi})\delta t} | \psi_{k-1}\rangle = \langle \psi_k | e^{-\frac{i}{\hbar}H(\psi^*, \psi)\delta t} |\psi_{k-1}\rangle$$

$\hat{H}$ = normal ordered.

$$= \underbrace{\langle \psi_k | \psi_{k-1}\rangle}\, e^{-\frac{i}{\hbar}H(\psi^*, \psi)\delta t}$$

H = Grassmann even.

$$= e^{\frac{1}{\hbar}\psi_k^*\psi_{k-1}}\, e^{-\frac{i}{\hbar}H(\psi^*, \psi)\delta t}$$

Campbell-Baker-Hausdorff

$$= e^{\frac{1}{\hbar}\left(\psi_k^*\psi_{k-1} - i H(\psi^*, \psi)\delta t\right)}\, e^{\frac{i}{\hbar}\underbrace{[\psi_k^*\psi_{k-1}, H(\psi_k, \psi_{k-1})]}_{=0}\delta t}$$

Grassmann even numbers commute.

Then

$$A_{\psi_f \leftarrow \psi_i} = \int \left(\prod_{k=1}^{N} \frac{d\psi_k^* d\psi_k}{\hbar} \underbrace{e^{-\frac{1}{\hbar}\psi_k^*\psi_k}} \right) \prod_{k=1}^{N+1} e^{\frac{1}{\hbar}(\psi_k^*\psi_{k-1} - i H\delta t)}$$

Grassmann-even no. commute

$$= \int \left(\prod \frac{d\psi^* d\psi}{\hbar} \right) e^{\frac{1}{\hbar}\sum_{k=1}^{N} -\psi_k^*\psi_k}\, e^{\frac{1}{\hbar}\sum_{k=1}^{N+1}(\psi_k^*\psi_{k-1} - iH\delta t)}$$

bring together: $-\psi_{N+1}^*\psi_{N+1}$ missing, so add back.

$$= \int \left(\prod \frac{d\psi^* d\psi}{\hbar} \right) e^{\frac{1}{\hbar}\sum_{k=1}^{N+1}\left[-\psi_k^*\psi_k + \psi_k^*\psi_{k-1} - iH\delta t\right] + \frac{1}{\hbar}\psi_f^*\psi_f}$$

$$= \int \left(\quad \right) e^{\frac{i}{\hbar}\sum_{k=1}^{N+1}\delta t\left[i\hbar\, \psi_k^*\left(\frac{\psi_k - \psi_{k-1}}{\delta t}\right) - H\right] + \frac{1}{\hbar}\psi_f^*\psi_f}$$

Take continuum limit:

$$\delta t \to 0, \quad N \to \infty; \qquad \sum_{k=1}^{N+1}\delta t = \int dt \qquad \frac{\psi_k - \psi_{k-1}}{\delta t} = \frac{d\psi}{dt}$$

Continuum limit

$$A_{\psi_f \leftarrow \psi_i} = \int_{\psi(t_i)=\psi_i}^{\psi(t_f)=\psi_f} \mathcal{D}\psi^* \mathcal{D}\psi \; e^{\frac{i}{\hbar}\underbrace{\int_{t_i}^{t_f} dt \left[i\psi^*\dot{\psi} - H \right]}_{\text{identify as action}} + \frac{1}{\hbar}\psi_f^*\psi_f}$$

So, finally:

$$A_{\psi_f \leftarrow \psi_i} = \langle \psi_f | e^{-\frac{i}{\hbar}H(t_f - t_i)} | \psi_i \rangle = \underbrace{e^{\frac{1}{\hbar}\psi_f^*\psi_f}} \int_{\psi_i}^{\psi_f} \mathcal{D}\psi^* \mathcal{D}\psi \; e^{\frac{i}{\hbar}S[\psi^*, \psi]}$$

Boundary term, (also appears for bosonic coherent path integration)
Just a constant, and doesn't affect correlation functions.

$$\mathcal{A} \longrightarrow \int_{\substack{\psi(t_f)=\psi_f \\ \psi(t_i)=\psi_i}} \mathcal{D}\psi^* \mathcal{D}\psi \; e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt \left[\underbrace{i\psi^*\dot{\psi} - H(\psi^*,\psi)}_{\text{identify as action.}}\right] + \psi_f^*\psi_f}$$

$$\mathcal{A}_{\psi_f \leftarrow \psi_i} = \int \mathcal{D}\psi^* \mathcal{D}\psi \; e^{iS[\psi^*,\psi]/\hbar + \psi_f^*\psi_f}$$

$$\langle \psi_f | e^{-\frac{i}{\hbar}H(t_f-t_i)} | \psi_i \rangle$$

this is a boundary term.
Just a constant, and does not affect correlation functions.

If I wish to compute Tr or STr,
I need to set $\psi_f = \mp\psi_i$ and integrate over ψ_f: $e^{+\psi_f^*\psi_f}$ is the necessary factor to make the coherent integrals possible.

$$\mathrm{Tr}\left[e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}\right] = \int d\psi_f^* \, d\psi_f \; \langle -\psi_f | e^{-\frac{i}{\hbar}H(t_f-t_i)} | \psi_i \rangle$$

$$= \int d\psi_f^* \, d\psi_f \; e^{\psi_f^*\psi_f} \int_{\psi_f = -\psi_i} \mathcal{D}\psi^* \mathcal{D}\psi \; e^{iS[\psi^*,\psi]/\hbar}$$

$$= \int_{\substack{\text{Anti} \\ \text{per. BC}}} \mathcal{D}\psi^* \mathcal{D}\psi \; e^{iS[\psi]/\hbar}$$

Similarly $\mathrm{STr}\left[e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}\right] = \int_{PBC} \mathcal{D}\psi^* \mathcal{D}\psi \, e^{iS[\psi]/\hbar}$.