Supersymmetric extension of the Poincaré algebra

Poincaré algebra:

$$[P_\mu, P_\nu] = 0$$

$$[M_{\mu\nu}, P_\rho] = i(g_{\nu\rho} P_\mu - g_{\mu\rho} P_\nu)$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(g_{\mu\rho} M_{\nu\sigma} - g_{\mu\sigma} M_{\nu\rho} - g_{\nu\rho} M_{\mu\sigma} + g_{\nu\sigma} M_{\mu\rho})$$

Take the above to form L_0. Build a $\mathbb{Z}_2$ graded algebra based on the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of Poincaré algebra, L_0.

Introduce Q_α, $\alpha = 1, 2$

$Q^{\dagger\dot{\alpha}}$ $\dot{\alpha} = \dot{1}, \dot{2}$

such that $[P^\mu, Q_\alpha] = 0$ $\qquad [P^\mu, Q^\dagger_{\dot{\alpha}}] = 0$

$$[M^{\mu\nu}, Q_\alpha] = -(S_L^{\mu\nu})_\alpha{}^\beta Q_\beta \qquad [M^{\mu\nu}, Q^{\dagger\dot{\alpha}}] = -(S_R^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} Q^{\dagger\dot{\beta}}$$

$$(S_L^{\mu\nu}) = \frac{i}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu) \equiv \sigma^{\mu\nu}$$

$$(S_R^{\mu\nu}) = \frac{i}{4}(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu) \equiv \bar{\sigma}^{\mu\nu}$$

Simply tells us that Q_α & $Q^{\dagger\dot{\beta}}$ transform like Weyl spinors.

So, need $L_1 \times L_1 \to L_0$ part of algebra:

Ansatz:

$$\{Q_\alpha, Q_\beta\} = A^\mu_{\alpha\beta} P_\mu + B^{\mu\nu}_{\alpha\beta} M_{\mu\nu} \quad \Rightarrow \quad \{Q^{\dagger\dot{\alpha}}, Q^{\dagger\dot{\beta}}\} = A^{\mu\dot{\alpha}\dot{\beta}} P_\mu + B^{\mu\nu\dot{\alpha}\dot{\beta}} M_{\mu\nu}$$

$$\{Q_\alpha, Q^\dagger_{\dot{\beta}}\} = E^\mu_{\alpha\dot{\beta}} P_\mu + F^{\mu\nu}_{\alpha\dot{\beta}} M_{\mu\nu}$$

symmetry properties: $A_{\alpha\beta} = A_{\beta\alpha}$ $\quad B^{\mu\nu} = -B^{\nu\mu}$

$B_{\alpha\beta} = B_{\beta\alpha}$ $\quad F^{\mu\nu} = -F^{\nu\mu}$

Determine A^μ, $B^{\mu\nu}$, E^μ, $F^{\mu\nu}$ using the Jacobi identities.

①
$$[P_\mu, \{Q_\alpha, Q^\dagger_{\dot\beta}\}] - \{Q^\dagger_{\dot\beta}, \cancelto{0}{[P_\mu, Q_\alpha]}\} + \{Q_\alpha, \cancelto{0}{[Q^\dagger_{\dot\beta}, P_\mu]}\} = 0$$

$$\therefore \quad [P_\mu, E^\nu_{\alpha\dot\beta} P_\nu + F^{\nu\rho}_{\alpha\dot\beta} M_{\nu\rho}] = 0$$

$$\therefore \quad F^{\nu\rho}_{\alpha\dot\beta}[P_\mu, M_{\nu\rho}] = F^{\nu\rho}_{\alpha\dot\beta}(-i)(g_{\rho\mu}P_\nu - g_{\nu\mu}P_\rho) = 0 \qquad \Rightarrow \boxed{F^{\mu\nu} = 0}$$

②
$$\underset{\text{remember } F=0}{[M^{\mu\nu}, \{Q_\alpha, Q^\dagger_{\dot\beta}\}]} - \{Q^\dagger_{\dot\beta}, [M^{\mu\nu}, Q_\alpha]\} + \{Q_\alpha, \underset{\epsilon_{\dot\beta\dot\gamma}[Q^{\dagger\dot\gamma}, M^{\mu\nu}]}{[Q^\dagger_{\dot\beta}, M^{\mu\nu}]}\} = 0$$

$$\therefore \quad [M^{\mu\nu}, E^\rho_{\alpha\dot\beta} P_\rho] - \{Q^\dagger_{\dot\beta}, -(S^{\mu\nu}_L)_\alpha{}^\gamma Q_\gamma\} + \{Q_\alpha, \epsilon_{\dot\beta\dot\gamma}(S^{\mu\nu}_R)^{\dot\gamma}{}_{\dot\delta}\, \underset{\nwarrow\ \epsilon^{\dot\delta\dot\epsilon}Q^\dagger_{\dot\epsilon}}{Q^{\dagger\dot\delta}}\} = 0$$

$$\therefore \quad E^\rho_{\alpha\beta}\, i(\delta^\nu{}_\rho P^\mu - \delta^\mu{}_\rho P^\nu) + (S^{\mu\nu}_L)_\alpha{}^\gamma E^\rho_{\gamma\beta} P_\rho + \epsilon_{\dot\beta\dot\gamma}(S^{\mu\nu}_R)^{\dot\gamma}{}_{\dot\delta}\, \underset{\sim\ominus}{\epsilon^{\dot\delta\dot\epsilon}}\, E^\rho_{\alpha\dot\epsilon} P_\rho = 0$$

$$\therefore \quad i(E^\nu_{\alpha\beta} g^{\mu\rho} - E^\mu_{\alpha\beta} g^{\nu\rho}) P_\rho + (S^{\mu\nu}_L)_\alpha{}^\gamma E^\rho_{\gamma\dot\beta} P_\rho - (S^{\mu\nu}_R)^{\dot\epsilon}{}_{\dot\beta} E^\rho_{\alpha\dot\epsilon} P_\rho = 0 \qquad \begin{pmatrix}\text{eqn } 2.63\\ \text{Martin}\end{pmatrix}$$

$$\therefore \quad E^\nu_{\alpha\dot\beta} g^{\mu\rho} - E^\mu_{\alpha\dot\beta} g^{\nu\rho} - i\left((S^{\mu\nu}_L)_\alpha{}^{\dot\gamma} E_{\gamma\dot\beta} - E^\rho_{\alpha\dot\epsilon}(S^{\mu\nu}_R)^{\dot\epsilon}{}_{\dot\beta}\right) = 0$$

(The P's are lin. indep.)

Solution: $\boxed{\text{Let } \; E^\mu_{\alpha\dot\beta} = \mathcal{N}\sigma^\mu_{\alpha\dot\beta}}$ $\quad \mathcal{N} =$ normalization of Q. $\equiv 2$ (convention)

Exercise: Show that this choice satisfies the Jacobi identity

Solution to Exercise

Solution: Let $\boxed{E^{\mu}_{\alpha\dot\beta} = \mathcal{N}\sigma^{\mu}_{\alpha\dot\beta}}$

$$\mathcal{N}\left[\sigma^{\nu}_{\alpha\dot\beta}\, g^{\mu\rho} - \sigma^{\mu}_{\alpha\dot\beta}\, g^{\nu\rho} - i\left((S_L^{\mu\nu})_{\alpha}{}^{\phi}\,\sigma^{\rho}_{\phi\dot\beta} - \sigma^{\rho}_{\alpha\dot\epsilon}\,(S_R^{\mu\nu})^{\dot\epsilon}{}_{\dot\beta}\right)\right] = 0$$

Everything is in proper matrix multiplication order — drop indices.

$$\sigma^{\nu} g^{\mu\rho} - \sigma^{\mu} g^{\nu\rho} - i\left(\frac{i}{4}(\sigma^{\mu}\bar\sigma^{\nu} - \sigma^{\nu}\bar\sigma^{\mu})\sigma^{\rho} - \sigma^{\rho}\frac{i}{4}(\bar\sigma^{\mu}\sigma^{\nu} - \sigma^{\nu}\sigma^{\mu})\right) = 0$$

$$'' \quad + \frac{1}{4}\left(\sigma^{\mu}\bar\sigma^{\nu}\sigma^{\rho} - \sigma^{\rho}\bar\sigma^{\mu}\sigma^{\nu} - (\mu\leftrightarrow\nu)\right)$$

(Martin eqn 2.43)

$$+ \frac{1}{4}\Big(g^{\mu\nu}\sigma^{\rho} - g^{\mu\rho}\sigma^{\nu} + g^{\nu\rho}\sigma^{\mu} + \cancel{i\epsilon^{\mu\nu\rho\lambda}}\sigma_{\lambda}$$
$$- g^{\rho\mu}\sigma^{\nu} + g^{\rho\nu}\sigma^{\mu} - g^{\mu\nu}\sigma^{\rho} - \cancel{i\epsilon^{\rho\mu\nu\lambda}}\sigma_{\lambda} - (\mu\leftrightarrow\nu)\Big)$$

$\oplus$

$$+ \frac{1}{4}2\left(g^{\mu\nu}\sigma^{\rho} - g^{\mu\rho}\sigma^{\nu} + g^{\nu\rho}\sigma^{\mu} - (\mu\leftrightarrow\nu)\right)$$

$$+ \frac{1}{4}2\Big(\cancel{g^{\mu\nu}\sigma^{\rho}} - g^{\mu\rho}\sigma^{\nu} + g^{\nu\rho}\sigma^{\mu}$$
$$- \cancel{g^{\nu\mu}\sigma^{\rho}} + g^{\nu\rho}\sigma^{\mu} - g^{\mu\rho}\sigma^{\nu}\Big)$$

$$\sigma^{\nu} g^{\mu\rho} - \sigma^{\mu} g^{\nu\rho} + \frac{1}{\cancel{4}}\cancel{2}\times 2\left(-g^{\mu\rho}\sigma^{\nu} + g^{\nu\rho}\sigma^{\mu}\right) = 0 \quad ✓$$

CANCEL

③ $$[P_\mu, \{Q_\alpha, Q_\beta\}] - \{Q_\beta, \overset{0}{\cancel{[P_\mu, Q_\alpha]}}\} + \{Q_\alpha, \overset{0}{\cancel{[Q_\beta, P_\mu]}}\} = 0$$

$$\therefore\ [P_\mu,\ A^\nu_{\alpha\beta} P_\nu + B^{\nu\rho}_{\alpha\beta} M_{\nu\rho}] = 0$$

$$\therefore\ B^{\nu\rho}_{\alpha\beta}\,[P_\mu, M_{\nu\rho}] = B^{\nu\rho}_{\alpha\beta}(-i)\underbrace{(g_{\rho\mu} P_\nu - g_{\nu\mu} P_\rho)}_{\text{not zero, in general}} = 0 \quad \Rightarrow \boxed{B^{\mu\nu}_{\alpha\beta} = 0}$$

④ $$[M^{\mu\nu}, \{Q_\alpha, Q_\beta\}] - \{Q_\beta, [M^{\mu\nu}, Q_\alpha]\} + \{Q_\alpha, [Q_\beta, M^{\mu\nu}]\} = 0$$

$$\therefore\ [M^{\mu\nu}, A^\rho_{\alpha\beta} P_\rho] - \{Q_\beta, -(S_L^{\mu\nu})_\alpha{}^\gamma Q_\gamma\} + \{Q_\alpha, (S_L^{\mu\nu})_\beta{}^\gamma Q_\gamma\} = 0$$

$$\therefore\ A^\rho_{\alpha\beta}\, i(\delta^\nu{}_\rho P^\mu - \delta^\mu{}_\rho P^\nu) + (S_L^{\mu\nu})_\alpha{}^\gamma A^\rho_{\underset{\leftrightarrow}{\beta\gamma}} P_\rho + (S_L^{\mu\nu})_\beta{}^\gamma A^\rho_{\underset{\leftrightarrow}{\alpha\gamma}} P_\rho = 0$$

$$\therefore\ i(A^\nu_{\alpha\beta}\, g^{\mu\rho} - A^\mu_{\alpha\beta}\, g^{\nu\rho}) P_\rho + \left((S_L^{\mu\nu})_\alpha{}^\gamma A^\rho_{\gamma\beta} + (S_L^{\mu\nu})_\beta{}^\gamma A^\rho_{\gamma\alpha}\right) P_\rho = 0$$

$$\Rightarrow\ A^\nu_{\alpha\beta}\, g^{\mu\rho} - A^\mu_{\alpha\beta}\, g^{\nu\rho} - i\left[(S_L^{\mu\nu})_\alpha{}^\gamma A^\rho_{\gamma\beta} + (S_L^{\mu\nu})_\beta{}^\gamma A^\rho_{\gamma\alpha}\right] = 0$$

Multiply by $g_{\nu\rho}$ and sum: contract ν, ρ.

$$A^\mu_{\alpha\beta} - 4A^\mu_{\alpha\beta} - i\left[(S_L^{\mu\nu})_\alpha{}^\gamma (A_\nu)_{\gamma\beta} + (S_L^{\mu\nu})_\beta{}^\gamma (A_\nu)_{\gamma\alpha}\right] = 0$$

$$A^\mu_{\alpha\beta} = \frac{-i}{3}\left[(S_L^{\mu\nu})_\alpha{}^\gamma (A_\nu)_{\gamma\beta} + (S_L^{\mu\nu})_\beta{}^\gamma (A_\nu)_{\gamma\alpha}\right]$$

$$\Rightarrow \boxed{A^\mu_{\alpha\beta} = 0}$$

Thus, the Super-Poincaré algebra is:

$$[P^\mu, P^\nu] = 0$$

$$[M^{\mu\nu}, P^\rho] = i(g^{\nu\rho}P^\mu - g^{\mu\rho}P^\nu)$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i(g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho})$$

$$[P^\mu, Q_\alpha] = 0$$

$$[P^\mu, \bar{Q}^{\dot\alpha}] = 0$$

$$[M^{\mu\nu}, Q_\alpha] = -(S_L^{\mu\nu})_\alpha{}^\beta Q_\beta$$

$$[M^{\mu\nu}, Q^{\dagger\dot\alpha}] = -(S_R^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} Q^{\dagger\dot\beta}$$

$$\{Q_\alpha, Q_\beta\} = 0$$

$$\{\bar{Q}^{\dot\alpha}, \bar{Q}^{\dot\beta}\} = 0$$

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta}P_\mu$$

	Canonical dimension	SU(2)×SU(2) rep
P^μ	1	$(1/2, 1/2)$
$M^{\mu\nu}$	0	$(1,0) \otimes (0,1)$
Q_α	1/2	$(1/2, 0)$
$Q^{\dagger\dot\alpha}$	1/2	$(0, 1/2)$

There are

$$L_0\left\{\begin{array}{l}\left.\begin{array}{l}-3\text{ generators of rotation, }\ J_i\\-3\text{ generators of boosts, }\ K_i\end{array}\right\}\ M^{\mu\nu}\qquad i=\{1,2,3\}\text{ each.}\\-4\text{ generators of translations}\qquad P^\mu\qquad \mu=\{0,1,2,3\}.\end{array}\right.$$

$$L_1\rightarrow -2\text{ generators}^*\text{ of SUSY transf.}\qquad Q_\alpha\ ,\quad \alpha=\{1,2\}.$$

12 generators in all.

* SUSY transformations take the form: $e^{i(\alpha^\alpha Q_\alpha + \alpha^\dagger_{\dot\alpha}Q^{\dagger\dot\alpha})}$ α^α and $\alpha^\dagger_{\dot\alpha}$ not indep.

- needed for unitarity