Superspace formalism

Supersymmetry is a spacetime symmetry. Extend spacetime by adding four anticommuting Grassman coordinates: $x^\mu \to (x^\mu, \theta_\alpha, \theta^\dagger_{\dot\beta})$ $\quad \alpha = \{1,2\}$, $\dot\beta = \{1,2\}$.

$$\{\theta_\alpha, \theta_\beta\} = \{\theta^\dagger_{\dot\alpha}, \theta^\dagger_{\dot\beta}\} = \{\theta_\alpha, \theta^\dagger_{\dot\beta}\} = 0$$

Mass dimensions

$[x_\mu] = -1$

$[\theta_\alpha] = -1/2$

$[\bar\theta_{\dot\alpha}] = -1/2$

So, superspace is eight-dimensional:

$$(\underbrace{x^0; x_1, x_2, x_3;}_{\text{Mix under rotations } J, \text{ and boosts } K.} \underbrace{\theta_1, \theta_2}_{\text{Mix under } J+iK}; \underbrace{\theta^\dagger_{\dot 1}, \theta^\dagger_{\dot 2}}_{\text{Mix under } J-iK.})$$

Using these extra coordinates, one can turn the graded Lie algebra into an ordinary Lie algebra:

$$[\theta^\alpha Q_\alpha, \theta^\dagger_{\dot\beta} Q^{\dagger\dot\beta}] = 2\theta^\alpha \sigma^\mu_{\alpha\dot\beta} \theta^{\dagger\dot\beta} P_\mu$$

$$[\theta^\alpha Q_\alpha, \theta^\beta Q_\beta] = 0$$

$$[\theta^\dagger_{\dot\alpha} Q^{\dagger\dot\alpha}, \theta^\dagger_{\dot\beta} Q^{\dagger\dot\beta}] = 0$$

Proof

$$[\theta^\alpha Q_\alpha, \bar\theta_{\dot\beta} \bar Q^{\dot\beta}] = \theta^\alpha Q_\alpha \theta^\dagger_{\dot\beta} Q^{\dagger\dot\beta} - \theta^\dagger_{\dot\beta} Q^{\dagger\dot\beta} \theta^\alpha Q_\alpha$$

raise/lower $\dot\beta$ $\Rightarrow -1$

pass $\theta^\dagger_{\dot b}$ through $Q^\dagger \theta Q$ $\Rightarrow -1$

pass θ^α through $Q^\dagger$ $\Rightarrow -1$

$\overline{\quad -1}$

$$= \theta^\alpha Q_\alpha Q^\dagger_{\dot\beta} \theta^{\dagger\dot\beta} + \theta^\alpha Q^\dagger_{\dot\beta} Q_\alpha \theta^{\dagger\dot\beta}$$

$$= \theta^\alpha \{Q_\alpha, Q^\dagger_{\dot\beta}\} \theta^{\dagger\dot\beta}$$

$$= 2\theta^\alpha \sigma^\mu_{\alpha\dot\beta} \theta^{\dagger\dot\beta} P_\mu$$

Calculus of Grassman variables

Differentials involving Grassman variables, $\theta_1, \theta_2, \ldots$ are defined by

$$d(\theta_1\theta_2\cdots\theta_n) = \sum_{i=1}^{n} \theta_1 \cdots d\theta_i \cdots \theta_n \qquad \text{(product rule)}$$

$$= \sum_i (-1)^{n-i}\, \theta_1 \cdots \hat{\theta}_i \cdots \theta_n \, d\theta_i$$

with $d\theta_i$'s placed to the right, so that differentiation with respect to a Grassman variable is conventionally defined to act from the right:

$$\frac{d}{d\theta_j}(\theta_1\cdots\theta_n)\, d\theta_j \equiv \theta_1\cdots\theta_n \qquad \text{(no sum)} \qquad \underline{\text{Definition}}$$

$$\Rightarrow \frac{d}{d\theta_j}(\theta_1\cdots\theta_n) = \underline{\underline{(-1)^{n-j}}}\, \theta_1 \cdots \hat{\theta}_j \cdots \theta_n . \qquad \text{(hatted } \theta \text{ means missing from product)}$$

NOTE MINUS SIGN!

On the other hand,

differentials with respect to complex conjugated Grassman variables are defined by

$$d(\theta_1^*\theta_2^*\cdots\theta_n^*) = \sum_{i=1}^{n} \theta_1^* \cdots d\theta_i^* \cdots \theta_n^*$$

$$= \sum_i (-1)^{i+1}\, d\theta_i^*\, \theta_1^* \cdots \hat{\theta}_i^* \cdots \theta_n^*$$

with the $d\theta_i^*$'s placed to the left, so that

$$\frac{d}{d\theta_j^*}\, d\theta_j^*\, (\theta_1^*\cdots\theta_n^*) \equiv \theta_1^*\cdots\theta_n^*$$

$$\Rightarrow \frac{d}{d\theta_j^*}(\theta_1^*\cdots\theta_n^*) = (-1)^{n}\, \theta_1^* \cdots \hat{\theta}_j^* \cdots \theta_n^*$$

So that differentiation with respect to complex conjugated Grassman variables act from the left.