Very helpful spinor identities for Grassmann coordinates

$$\theta_\alpha \theta_\beta = \frac{1}{2}\epsilon_{\alpha\beta}(\theta\theta)$$

$$\theta^\dagger_{\dot\alpha}\theta^\dagger_{\dot\beta} = -\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}(\theta^\dagger\theta^\dagger)$$

$$\theta_\alpha \theta^\dagger_{\dot\beta} = \frac{1}{2}\sigma^\mu_{\alpha\dot\beta}(\theta\sigma_\mu\theta^\dagger)$$

Then some helpful Fierz identities can be derived:

$$(\theta\xi)(\theta\chi) = -\frac{1}{2}(\theta\theta)(\xi\chi) \qquad (\theta^\dagger\xi^\dagger)(\theta^\dagger\chi^\dagger) = -\frac{1}{2}(\theta^\dagger\theta^\dagger)(\xi^\dagger\chi^\dagger)$$

$$(\theta\xi)(\theta^\dagger\chi^\dagger) = \frac{1}{2}(\theta\sigma^\mu\theta)(\xi\sigma_\mu\chi^\dagger)$$

$$\theta^\dagger\bar\sigma^\mu\theta = \underbrace{-\theta\sigma^\mu\theta^\dagger = (\theta^\dagger\bar\sigma^\mu\theta)^*}_{\text{satisfies "reality condition"}}$$

$$(\theta\sigma^\mu\theta^\dagger)(\theta\sigma^\nu\theta^\dagger) = \frac{1}{2}g^{\mu\nu}(\theta\theta)(\theta^\dagger\theta^\dagger)$$

$$\theta\sigma^\mu\bar\sigma^\nu\theta = g^{\mu\nu}(\theta\theta) \quad \text{and} \quad \theta^\dagger\bar\sigma^\mu\sigma^\nu\theta = g^{\mu\nu}(\theta^\dagger\theta^\dagger)$$

$$\Rightarrow \quad \theta\sigma^{\mu\nu}\theta = \theta^\dagger\bar\sigma^{\mu\nu}\theta^\dagger = 0.$$

Notation:

Differentiation with respect to superspace coordinates: θ_α & $\theta^{\dagger\dot\alpha}$

$$\partial^\alpha \equiv \frac{\partial}{\partial\theta_\alpha} \qquad \partial_\alpha \equiv \frac{\partial}{\partial\theta^\alpha}$$

$$\partial^\dagger_{\dot\alpha} \equiv \frac{\partial}{\partial\theta^{\dagger\dot\alpha}} \qquad \partial^{\dagger\dot\alpha} \equiv \frac{\partial}{\partial\theta^\dagger_{\dot\alpha}}$$

n.b.: no minus sign, unlike $\left(\frac{\partial}{\partial t}\right)^\dagger = -\frac{\partial}{\partial t}$

Definition of derivatives:

$$\partial_\alpha \theta^\beta = \frac{\partial}{\partial\theta^\alpha}\theta^\beta = \delta_\alpha{}^\beta$$

$$\partial_\alpha \theta_\beta = \frac{\partial}{\partial\theta^\alpha}\theta_\beta = \epsilon_{\beta\gamma}\frac{\partial}{\partial\theta^\alpha}\theta^\gamma = \epsilon_{\beta\gamma}\delta_\alpha{}^\gamma = \underline{\epsilon_{\beta\alpha} = -\epsilon_{\alpha\beta}}$$

$$\partial^\dagger_{\dot\alpha}\theta^{\dagger\dot\beta} = \frac{\partial}{\partial\theta^{\dagger\dot\alpha}}\theta^{\dagger\dot\beta} = \delta_{\dot\alpha}{}^{\dot\beta}$$

$$\partial^\dagger_{\dot\alpha}\theta^\dagger_{\dot\beta} = \frac{\partial}{\partial\theta^{\dagger\dot\alpha}}\theta^{\dagger\dot\beta} = \epsilon_{\dot\beta\dot\gamma}\frac{\partial}{\partial\theta^{\dagger\dot\alpha}}\theta^{\dagger\dot\gamma} = \epsilon_{\dot\beta\dot\gamma}\delta_{\dot\alpha}{}^{\dot\gamma} = \underline{\epsilon_{\dot\beta\dot\alpha} = -\epsilon_{\dot\alpha\dot\beta}}$$

Note the extra minus sign! Implies raising and lowering indices on ∂_α and $\partial^\dagger_{\dot\alpha}$ results in an extra minus sign.

$$\epsilon^{\alpha\beta}\partial_\beta = -\partial^\alpha \qquad \partial_\alpha = -\epsilon_{\alpha\beta}\partial^\beta$$

$$\epsilon^{\dot\alpha\dot\beta}\partial^\dagger_{\dot\beta} = -\partial^{\dagger\dot\alpha} \qquad \partial^\dagger_{\dot\alpha} = -\epsilon_{\dot\alpha\dot\beta}\partial^{\dagger\dot\beta}$$

Derivatives anticommute with themselves (as ordinary derivatives commute)

$$\{\partial_\alpha, \partial_\beta\} = 0 \quad \text{and} \quad \{\partial^{\dagger\dot\alpha}, \partial^{\dagger\dot\beta}\} = 0$$

also

$$\{\partial_\alpha, \theta^\beta\} = \{\theta^\beta, \partial_\alpha\} = \delta_\alpha{}^\beta$$

$$\{\partial^\dagger_{\dot\alpha}, \theta^{\dagger\dot\beta}\} = \{\theta^{\dagger\dot\beta}, \partial^\dagger_{\dot\alpha}\} = \delta_{\dot\alpha}{}^{\dot\beta}$$

$$\frac{\partial}{\partial\theta^\alpha}(\psi\theta) = \frac{\partial}{\partial\theta^\alpha}\psi^\beta\theta_\beta = \frac{\partial}{\partial\theta^\alpha}\theta^\beta\psi_\beta = \delta_\alpha{}^\beta\psi_\beta = \psi_\alpha$$

<u>compact form:</u> $\partial_\alpha(\psi\theta) = \psi_\alpha$

<u>Integration over superspace:</u> — to be distinguished from functional integral over <u>spinors</u> which are <u>Gaussian</u> in nature.

$$d^2\theta = -\frac{1}{4}d\theta^\alpha d\theta^\beta\epsilon_{\alpha\beta} = -\frac{1}{4}d\theta^1 d\theta^2\underbrace{\epsilon_{12}}_{-1} - \frac{1}{4}d\theta^2 d\theta^1\underbrace{\epsilon_{21}}_{+1}$$

$$= -\frac{1}{2}d\theta^2 d\theta^1$$

$$d^2\theta^\dagger = -\frac{1}{4}d\theta^\dagger_{\dot\alpha}d\theta^\dagger_{\dot\beta}\epsilon^{\dot\alpha\dot\beta} = -\frac{1}{4}d\theta^\dagger_{\dot 1}d\theta^\dagger_{\dot 2}\underbrace{\epsilon^{\dot 1\dot 2}}_{+1} - \frac{1}{4}d\theta^\dagger_{\dot 2}d\theta^\dagger_{\dot 1}\underbrace{\epsilon^{\dot 2\dot 1}}_{-1}$$

$$= +\frac{1}{2}d\theta^\dagger_{\dot 2}d\theta^\dagger_{\dot 1}$$

<u>Reason for normalization:</u>

$$\int d^2\theta\ \theta\theta = \int -\frac{1}{2}d\theta^2 d\theta^1\underbrace{\left[\theta^1\theta_1 + \theta^2\theta_2\right]}_{\theta^2\theta^1 - \theta^1\theta^2}$$

$$= \int -\frac{1}{2}d\theta^2 d\theta^1(-2\theta^1\theta^2) = \left(-\frac{1}{2}\right)(-1)\,1$$

similarly,

$$\int d^2\theta^\dagger(\theta^\dagger\theta^\dagger) = \int \frac{1}{2}d\theta^\dagger_{\dot 2}d\theta^\dagger_{\dot 1}\left[\theta^\dagger_{\dot 1}\theta^{\dagger\dot 2} + \theta^\dagger_{\dot 2}\theta^{\dagger\dot 1}\right]$$

$$= \int \frac{1}{2}d\theta^\dagger_{\dot 2}d\theta^\dagger_{\dot 1}\left[2\,\theta^\dagger_{\dot 1}\theta^\dagger_{\dot 2}\right] = \left(\frac{1}{2}\right)(2) = 1\ \checkmark$$