Graded Lie Algebra

Is a graded algebra with the following additional properties:

For $\mathbb{Z}_2$-graded Lie algebra

① Grading (obvious)

For all $a_i \in L_i \quad (i = 0, 1)$.

$$x_i \circ x_j \in L_{i+j,\ \mathrm{mod}\ 2}$$

② Supersymmetrization: (replaces antisymmetry)

For all $x_i \in L_i, \quad x_j \in L_j, \quad (i, j = 0, 1)$.

$$x_i \circ x_j = -(-1)^{i \cdot j}\, x_j \circ x_i$$

③ Generalized Jacobi Identities:

For all $x_i \in L_i,\ x_j \in L_j,\ x_k \in L_k, \quad (i, j, k = 0, 1)$:

$$(-1)^{ik} x_i \circ (x_j \circ x_k) + (-1)^{ij} x_j \circ (x_k \circ x_i) + (-1)^{jk} x_k \circ (x_i \circ x_j) = 0$$

Although the set of integers form a $\mathbb{Z}_2$ graded algebra under addition, it does not form a $\mathbb{Z}_2$ graded Lie algebra because it does not obey supersymmetrization; i.e. $(1+3) \neq -(3+1)$

Graded Lie algebra of $SU(N)$

$$L = \underset{\text{evens}}{L_0} \oplus \underset{\text{odds}}{L_1}, \qquad L \times L \to L$$

Start with $L_0 = SU(N)$ algebra T^a, $a = \{1, 2, \ldots N^2-1\}$

$$T^a \circ T^b = [T^a, T^b] = i f^{abc} T^c$$

Now define L_1: Q_i, $i = \{1, 2, \ldots n\}$ $n \equiv$ dimension of L_1.

Need to define product $\circ$: $\left\{ \begin{array}{l} T^a \circ Q_i \longrightarrow L_1 \\ Q_i \circ Q_j \longrightarrow L_0 \end{array} \right.$

Define $T^a \circ Q_i = -c^a_{ij} Q_j$ (minus sign for convenience)

The c^a_{ij} are coefficients to be determined by the Jacobi identity for T^a, T^b, Q_i:

$$(-1)^0 T^a \circ (T^b \circ Q_i) + (-1)^0 T^b \circ (Q_i \circ T^a) + (-1)^0 Q_i \circ (T^a \circ T^b) = 0$$

$$T^a \circ (-c^b_{ij}) Q_j + T^b \circ c^a_{ij} Q_j + Q_i \circ i f^{abc} T^c = 0$$

$$c^b_{ij} c^a_{jk} Q_k + c^a_{ij}(-c^b_{jk}) Q_k + i f^{abc} c^c_{ik} Q_k = 0$$

or, because Q_k are independent,

$$\boxed{[c^a, c^b]_{ik} = i f^{abc} c^c_{ik}}$$

Hence, c^a_{ij} matrices provide an $n \times n$ matrix representation of L_0. Therefore, to construct a $\mathbb{Z}_2$ graded Lie algebra, we must pick a representation for L_0, and the number of Q_i's matches the dimension of the representation.

Define: $Q_i \circ Q_j = \left(+Q_j \circ Q_i\right) = \underset{\substack{\uparrow \\ \text{symmetric in } i \leftrightarrow j.}}{d^a_{ij}}\, T^a$

Consider the Jacobi identity for T^a, Q_i, Q_j:

$$(-1)^0\, T^a \circ (Q_i \circ Q_j) + (-1)^0\, Q_i \circ (Q_j \circ T^a) + (-1)^1 Q_j \circ (T^a \circ Q_i) = 0$$

$$T^a \circ d^c_{ij}\, T^c + Q_i \circ c^a_{jk} Q_k - Q_j \circ (-c^a_{ik}) Q_k = 0$$

$$d^c_{ij}\, \underset{\ominus}{if^{acb}}\, T^b + c^a_{jk}\, \underset{\oplus}{d^b_{ik}}\, T^b + c^a_{ik}\, \underset{\oplus}{d^b_{jk}}\, T^b = 0$$

or, because the T^b are independent

$$\boxed{c^a_{ik}\, d^b_{kj} + c^a_{jk}\, d^b_{ki} = if^{abc}\, d^c_{ij}} \qquad *$$

Once c^a_{ij} are known (are representation matrices), can construct the d^a_{ij} by considering the above as a system of linear equations.

Example $\mathbb{Z}_2$ graded $SU(2)$; based on the fundamental representation:

– choose representation: 2 (doublet □) $f^{abc} = \epsilon^{abc}$

Immediately, $\boxed{c^1 = \frac{1}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad c^2 = \frac{1}{2}\begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad c^3 = \frac{1}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}}$

then let $d^1 = \begin{pmatrix} d^1_1 & d^1_x \\ d^1_x & d^1_2 \end{pmatrix}$, $d^2 = \begin{pmatrix} d^2_1 & d^2_x \\ d^2_x & d^2_2 \end{pmatrix}$, $d^3 = \begin{pmatrix} d^3_1 & d^3_x \\ d^3_x & d^3_2 \end{pmatrix}$,

and solve the system of equations in (*) [Mathematica]

Result: $\boxed{d^1 = \mathcal{N}\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, \quad d^2 = \mathcal{N}\begin{pmatrix} i & \\ & i \end{pmatrix}, \quad d^3 = \mathcal{N}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$, $\mathcal{N}$ sets the normalization of Q's.

Hence our graded algebra is: (there are five elements, T^1, T^2, T^3, Q_1, Q_2)

$$\left.\begin{aligned} T^a \circ T^b &= i\epsilon^{abc}\, T^c \\ T^a \circ Q_i &= -\left(\frac{\sigma^a}{2}\right) Q_j \\ Q_i \circ Q_j &= \left(d^a_{ij}\right) T^a \end{aligned}\right\} \quad OSp(1/2) \quad \text{"one slash two"}$$

(Now, T^a, Q_i's can be any rep)

As a consistency checks, the Jacobi identity for three Q's must be satisfied:

$$(-1)^1 \, Q_i \circ (Q_j \circ Q_k) + (-1)^1 Q_j \circ (Q_k \circ Q_i) + (-1)^1 Q_k \circ (Q_i \circ Q_j) = 0$$

$$Q_i \circ d^a_{jk} T^a + Q_j \circ d^a_{ki} T^a + Q_k \circ d^a_{ij} T^a = 0$$

$$d^a_{jk} c^a_{i\ell} Q_\ell + d^a_{ki} c^a_{j\ell} Q_\ell + d^a_{ij} c^a_{k\ell} Q_\ell = 0$$

or, because Q_ℓ are indep,

$$\boxed{d^a_{jk} c^a_{i\ell} + d^a_{ki} c^a_{j\ell} + d^a_{ij} c^a_{k\ell} = 0}$$

For $OSp(1/2)$ checked on Mathematica.

■ Determining d_{ij}^{a}

In[42]:=
$$c = \left\{\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\};$$
$$d = \left\{\begin{pmatrix} d11 & d1x \\ d1x & d12 \end{pmatrix}, \begin{pmatrix} d21 & d2x \\ d2x & d22 \end{pmatrix}, \begin{pmatrix} d31 & d3x \\ d3x & d32 \end{pmatrix}\right\};$$

In[44]:=
```
sol = Solve[{c[[1]].d[[2]] + Transpose[c[[1]].d[[2]]] == ⅈ d[[3]],
    c[[2]].d[[3]] + Transpose[c[[2]].d[[3]]] == ⅈ d[[1]],
    c[[3]].d[[1]] + Transpose[c[[3]].d[[1]]] == ⅈ d[[2]]}]
```

Out[44]=
```
{{d11 → -d3x, d21 → ⅈ d3x, d12 → d3x, d22 → ⅈ d3x, d2x → 0, d31 → 0, d32 → 0, d1x → 0}}
```

In[45]:=
$$\begin{pmatrix} d11 & d1x \\ d1x & d12 \end{pmatrix} \text{ /. sol[[1]] // MatrixForm}$$
$$\begin{pmatrix} d21 & d2x \\ d2x & d22 \end{pmatrix} \text{ /. sol[[1]] // MatrixForm}$$
$$\begin{pmatrix} d31 & d3x \\ d3x & d32 \end{pmatrix} \text{ /. sol[[1]] // MatrixForm}$$

Out[45]//MatrixForm=
$$\begin{pmatrix} -d3x & 0 \\ 0 & d3x \end{pmatrix}$$

Out[46]//MatrixForm=
$$\begin{pmatrix} i\, d3x & 0 \\ 0 & i\, d3x \end{pmatrix}$$

Out[47]//MatrixForm=
$$\begin{pmatrix} 0 & d3x \\ d3x & 0 \end{pmatrix}$$

■ Consistency Check

In[48]:=
$$c = \left\{\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\};$$
$$d = \left\{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\};$$

In[50]:=
$$\text{sum[i\_, j\_, k\_, l\_] := } \sum_{a=1}^{3} (\text{d[[a, j, k]]} * \text{c[[a, i, l]]} + \text{d[[a, k, i]]} * \text{c[[a, j, l]]} + \text{d[[a, i, j]]} * \text{c[[a, k, l]]})$$

In[52]:=
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Table[sum[i, j, k, l], {i, 1, 2}, {j, 1, 2}, {k, 1, 2}, {l, 1, 2}]
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Out[52]=
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{{{{0, 0}, {0, 0}}, {{0, 0}, {0, 0}}}, {{{0, 0}, {0, 0}}, {{0, 0}, {0, 0}}}}
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More on $\mathbb{Z}_2$ graded Lie algebras

Generalized structure constants.

Let $L = L_0 \oplus L_1$ endowed with $\circ$ be our graded Lie Algebra.

$X_i \in L$ are elements of L.

$$X_i \circ X_j = c_{ij}^{k} X_k \qquad c_{ij}^{k} = \text{generalized structure constants.}$$

$$c_{ij}^{k} = -c_{ji}^{k} \quad \text{if} \quad \underset{\text{even-even}}{X_i, X_j \in L_0} \quad \text{or} \quad \underset{\text{even-odd}}{X_i \in L_0 \;\; X_j \in L_1}$$

$$c_{ij}^{k} = +c_{ji}^{k} \quad \text{if} \quad \underset{\text{odd-odd}}{X_i, X_j \in L_1}$$

Jacobi identity. Let $T^a \in L_0$ $Q_i \in L_1$:

$$[T^a, [T^b, T^c]] + [T^c, [T^a, T^b]] + [T^b, [T^c, T^a]] = 0$$

$$[T^a, [T^b, Q_i]] + [Q_i, [T^a, T^b]] + [T^b, [Q_i, T^a]] = 0$$

$$[T^a, \{Q_i, Q_j\}] \underset{\text{minus!}}{-} \{Q_j, [T^a, Q_i]\} + \{Q_i, [Q_j, T^a]\} = 0$$

$$[Q_i, \{Q_j, Q_k\}] + [Q_k, \{Q_i, Q_j\}] + [Q_j, \{Q_k, Q_i\}] = 0.$$

Killing form