ACTIVE supersymmetry transformations:

- generated by a unitary operator that acts on Hibert space:

$$\hat{U}(a_\mu, \alpha_a, \alpha^{\dagger\dot{a}}) = e^{+ia_\mu \hat{P}^\mu + i(\alpha\hat{Q} + \alpha^\dagger \hat{Q}^\dagger)}$$

c.f. $\hat{U}(\omega) = e^{-\frac{i}{2}\omega_{\mu\nu}\hat{M}^{\mu\nu}}$ for active rotations.

a_μ, α, $\alpha^\dagger$ are the 8 parameters of SUSY transformations.

Consider the (left) transformation of an arbitrary coset element $\Omega(x, \theta, \theta^\dagger) \equiv U(x, \theta, \theta^\dagger)$

$$e^{ia_\mu P^\mu + i(\alpha Q + \alpha^\dagger Q^\dagger)}\,\Omega(x, \theta, \theta^\dagger)$$

$$\longrightarrow e^{ia_\mu P^\mu + i(\alpha Q + \alpha^\dagger Q^\dagger)}\, e^{ix_\mu P^\mu + i(\theta Q + \theta^\dagger Q^\dagger)}$$

Use Baker-Campbell-Hausdorff formula to combine:

$$e^A e^B = e^{A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] - \frac{1}{12}[B,[B,A]] + \dots}$$

$$A = ia.P + i\alpha Q + i\alpha^\dagger Q^\dagger$$
$$B = ix.P + i\theta Q + i\theta^\dagger Q^\dagger$$

$$[A,B] = 0 + i^2[\alpha Q + \alpha^\dagger Q^\dagger,\ \theta Q + \theta^\dagger Q^\dagger]$$

$$= -\left([\alpha Q, \theta Q] + [\alpha Q, \theta^\dagger Q^\dagger] + [\alpha^\dagger Q^\dagger, \theta Q] + [\alpha^\dagger Q^\dagger, \theta^\dagger Q^\dagger]\right)$$

$$= -\left(0 + 2\alpha\sigma^\mu\theta^\dagger P_\mu - 2\theta\sigma^\mu\alpha^\dagger P_\mu + 0\right)$$

and $[A,[A,B]] = [B,[B,A]] = 0$, since $[P_\mu, Q^{(\dagger)}] = 0$.

Thus,

$$U(a, \alpha, \alpha^\dagger)\,\Omega(x, \theta, \theta^\dagger) = e^{ia.P + i\alpha Q + i\alpha^\dagger Q^\dagger + ix.P + i\theta Q + i\theta^\dagger Q^\dagger - \frac{1}{2}(2\alpha\sigma^\mu\theta^\dagger P_\mu - 2\theta\sigma^\mu\alpha^\dagger P_\mu)}$$

$$= e^{+i[x^\mu + a^\mu - i(\theta\sigma^\mu\alpha^\dagger - \alpha\sigma^\mu\theta^\dagger)]P_\mu - i(\theta + \alpha)Q - i(\theta^\dagger + \alpha^\dagger)Q^\dagger}$$

$$= \Omega\left(x^\mu + a^\mu - i(\theta\sigma^\mu\alpha^\dagger - \alpha\sigma^\mu\theta^\dagger),\ \theta + \alpha,\ \theta^\dagger + \alpha^\dagger\right)$$

Thus, a SUSY transformation (without spatial rotations & boosts) corresponds to translations in supercoordinates $(x;\theta,\bar\theta)$

$$\boxed{\begin{pmatrix} x^\mu \\ \theta \\ \bar\theta \end{pmatrix} \xrightarrow{\{a^\mu,\alpha,\bar\alpha\}} \begin{pmatrix} (x+a)^\mu - i(\theta\sigma^\mu\bar\alpha - \alpha\sigma^\mu\bar\theta) \\ \theta+\alpha \\ \bar\theta+\bar\alpha \end{pmatrix}}$$

Expect: when actively transforming a superfield:

$$\hat\Phi(x,\theta,\bar\theta) \longrightarrow \hat U(a,\alpha,\bar\alpha)\,\hat\Phi(x,\theta,\bar\theta)\,\hat U^\dagger(a,\alpha,\bar\alpha) = \left[e^{+i\alpha G(\hat Q)}\right]\hat\Phi(x,\theta,\bar\theta) \qquad (*)$$

$$= \hat\Phi\big(x-a+i(\theta\sigma\bar\alpha - \alpha\sigma\bar\theta),\ \theta-\alpha,\ \bar\theta-\bar\alpha\big) \qquad (**)$$

(minus signs for active transformations)

$$\xrightarrow{\text{to yield}} \left[e^{+i\alpha G(\hat Q)}\right]\Phi(x,\theta,\bar\theta)$$

$$(*) = e^{+ia_\mu G(\hat P^\mu) + i(\alpha G(\hat Q) + \bar\alpha G(\hat{\bar Q}))}\,\hat\Phi(x,\theta,\bar\theta)$$

$$\simeq \hat\Phi + ia_\mu G(\hat P^\mu) + i\alpha^a G(\hat Q_a) + i\bar\alpha_{\dot a} G(\hat{\bar Q}^{\dot a}) + \mathcal{O}(a^2)$$

$$(**) \simeq \Phi(x,\theta,\bar\theta) - \big(a - i(\theta\sigma\bar\alpha - \alpha\sigma\bar\theta)\big)_\mu\partial^\mu\Phi - \alpha^\alpha\frac{\partial}{\partial\theta^\alpha}\Phi - \bar\alpha_{\dot\alpha}\frac{\partial}{\partial\bar\theta_{\dot\alpha}}\Phi$$

collect $a,\alpha,\bar\alpha$:

$$= \Phi - a_\mu\partial^\mu\Phi - i\alpha^a\sigma^\mu_{a\dot b}\bar\theta^{\dot b}\partial_\mu - \alpha^a\frac{\partial}{\partial\theta^a}\Phi - i\bar\alpha_{\dot a}\bar\sigma^{\mu\dot a b}\theta_b\partial_\mu - \bar\alpha_{\dot a}\frac{\partial}{\partial\bar\theta_{\dot a}}\Phi$$

$$= \Phi + ia_\mu(i\partial^\mu)\Phi + i\alpha^a\left(-\sigma^\mu_{a\dot b}\bar\theta^{\dot b}\partial_\mu + i\frac{\partial}{\partial\theta^a}\right)\Phi + i\bar\alpha_{\dot a}\left(-\bar\sigma^{\mu\dot a b}\theta_b\partial_\mu + i\frac{\partial}{\partial\bar\theta_{\dot a}}\right)\Phi$$

Match:

$$G(\hat P^\mu) = i\partial^\mu$$

$$G(Q_a) = i\frac{\partial}{\partial\theta^a} - \sigma^\mu_{a\dot b}\bar\theta^{\dot b}\partial_\mu \qquad G(Q^a) = -i\frac{\partial}{\partial\theta_a} + \bar\theta_{\dot b}\bar\sigma^{\mu\dot b a}\partial_\mu$$

$$G(\bar Q^{\dot a}) = i\frac{\partial}{\partial\bar\theta_{\dot a}} - \bar\sigma^{\mu\dot a b}\theta_b\partial_\mu \qquad G(\bar Q_{\dot a}) = -i\frac{\partial}{\partial\bar\theta^{\dot a}} + \theta^b\sigma^\mu_{b\dot a}\partial_\mu$$

Alternate forms

$$\epsilon^{ca} Q_a = i\epsilon^{ca}\partial_a - \epsilon^{ca}\sigma^{\mu}_{a\dot{b}}\bar{\theta}^{\dot{b}}\partial_\mu$$

$$= -i\partial^c - \epsilon^{ca}\epsilon_{ad}\epsilon_{\dot{b}\dot{e}}\bar{\sigma}^{\mu\dot{e}d}\bar{\theta}^{\dot{b}}\partial_\mu$$

$$= -i\partial^c + \underbrace{\epsilon^{ca}\epsilon_{ad}}_{\delta^c{}_d}\bar{\theta}_{\dot{e}}\bar{\sigma}^{\mu\dot{e}d}\partial_\mu$$

$$Q^c = -i\partial^c + \bar{\theta}_{\dot{e}}\bar{\sigma}^{\mu\dot{e}c}\partial_\mu = -i\partial + \bar{\theta}\bar{\sigma}^{\mu}\partial_\mu$$

$$\epsilon_{\dot{c}\dot{a}}\bar{Q}^{\dot{a}} = i\epsilon_{\dot{c}\dot{a}}\bar{\partial}^{\dot{a}} - \epsilon_{\dot{c}\dot{a}}\bar{\sigma}^{\mu\dot{a}b}\theta_b\partial_\mu$$

$$= -i\bar{\partial}_{\dot{c}} - \epsilon_{\dot{c}\dot{a}}\epsilon^{\dot{a}\dot{d}}\epsilon^{be}\sigma^{\mu}_{e\dot{d}}\theta_b\partial_\mu$$

$$= -i\bar{\partial}_{\dot{c}} + \delta_{\dot{c}}{}^{\dot{d}}\theta^e\sigma^{\mu}_{e\dot{d}}\partial_\mu$$

$$\bar{Q}_{\dot{c}} = -i\bar{\partial}_{\dot{c}} + \theta^e\sigma^{\mu}_{e\dot{c}}\partial_\mu = -i\bar{\partial} + \theta\sigma^{\mu}\partial_\mu$$

⇒ Canonical differential representations are:

$$\boxed{\begin{aligned} G(P^\mu) &= i\partial^\mu \equiv \left(-i\frac{\partial}{\partial t}, -i\vec{\nabla}\right) \\ G(Q_a) &= i\partial_a - \sigma^{\mu}_{a\dot{b}}\bar{\theta}^{\dot{b}}\partial_\mu \\ G(\bar{Q}_{\dot{a}}) &= -i\bar{\partial}_{\dot{a}} + \theta^b\sigma^{\mu}_{b\dot{a}}\partial_\mu \end{aligned}}$$

as expected.
(see Peskin and Schröder)

Checking the SUSY algebra with differential operators:

Apply $\{Q_\alpha, Q^\dagger_{\dot\delta}\}$ to a test (super) function

$$\{Q_\alpha, Q^\dagger_{\dot\delta}\} = \left\{ i\frac{\partial}{\partial\theta^\alpha} - \sigma^\mu_{\alpha\dot\beta}\,\theta^{\dagger\dot\beta}\,\partial_\mu ,\ -i\frac{\partial}{\partial\theta^{\dagger\dot\delta}} + \theta^\gamma\sigma^\mu_{\gamma\dot\delta}\,\partial_\mu \right\}$$

$$= \underbrace{\left\{\frac{\partial}{\partial\theta^\alpha}, \frac{\partial}{\partial\theta^{\dagger\dot\delta}}\right\}}_{0} + i\sigma^\mu_{\gamma\dot\delta}\underbrace{\left\{\frac{\partial}{\partial\theta^\alpha}, \theta^\gamma\right\}}_{\delta_\alpha{}^\gamma}\partial_\mu + i\sigma^\mu_{\alpha\dot\beta}\underbrace{\left\{\theta^{\dagger\dot\beta}, \frac{\partial}{\partial\theta^{\dagger\dot\delta}}\right\}}_{\delta^{\dot\beta}_{\dot\delta}}\partial_\mu - \sigma^\mu_{\alpha\dot\beta}\sigma^\nu_{\gamma\dot\delta}\underbrace{\{\theta^{\dagger\dot\beta}, \theta^\gamma\}}_{0}\partial_\mu\partial_\nu$$

$$= i\sigma^\mu_{\alpha\dot\delta}\,\partial_\mu + i\sigma^\mu_{\alpha\dot\delta}\,\partial_\mu$$

$$= 2\sigma^\mu_{\alpha\dot\delta}(i\partial_\mu) = 2\sigma^\mu_{\alpha\dot\delta}P_\mu \quad ✓$$

Notice, if there were a relative minus sign between the two cross terms, anticommutator would vanish. This can be arranged by flipping sign of second terms in Q or $Q^\dagger$.

Let
$$iD_\alpha = i\frac{\partial}{\partial\theta^\alpha} + \sigma^\mu_{\alpha\dot\beta}\,\theta^{\dagger\dot\beta}\,\partial_\mu$$
$$iD^\dagger_{\dot\alpha} = -i\frac{\partial}{\partial\theta^{\dagger\dot\alpha}} - \theta^\beta\sigma^\mu_{\beta\dot\alpha}\,\partial_\mu$$
chiral covariant derivatives.

Then,

$$\{D_\alpha, Q_\beta\} = \{D_\alpha, Q^\dagger_{\dot\beta}\} = 0$$

$$\{D^\dagger_{\dot\alpha}, Q_\beta\} = \{D^\dagger_{\dot\alpha}, Q^\dagger_{\dot\beta}\} = 0$$

- more on this later.