$\boxed{\text{F-terms involving chiral superfield}}$

As in SUSY-QM,

① $\begin{pmatrix}\text{chiral}\\ \text{superfield}\end{pmatrix} + \begin{pmatrix}\text{chiral}\\ \text{superfield}\end{pmatrix} = \begin{pmatrix}\text{chiral}\\ \text{superfield}\end{pmatrix}$

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Thus, a good F-term candidate is:

$$S = \int d^4x \begin{bmatrix}\text{any analytic* function}\\ \text{of chiral superfields}\end{bmatrix}_F$$

* Terminology: more common is "Holomorphic"

$$= \int d^4x \left(\left[W(\Phi_i) \right]_F + c.c. \right)$$

$W \equiv$ superpotential (of many chiral superfields)

$$W(\Phi) = W(\phi) + (\Phi - \phi)\frac{\partial W}{\partial \phi} + \frac{1}{2}(\Phi-\phi)(\Phi-\phi)\frac{\partial^2 W}{\partial \phi^2} + \dots$$

Using $(y^\mu, \theta, \theta^\dagger)$ coordinates, series terminates after three terms. $\Phi = \phi + \sqrt{2}\theta\chi + \theta\theta F$

$$(\Phi - \phi) = \theta\sqrt{2}\chi + \theta\theta F$$

$$(\Phi-\phi)(\Phi-\phi) = (\theta\sqrt{2}\chi)(\theta\sqrt{2}\chi) = 2(\theta\chi)(\theta\chi) = -(\theta\theta)\chi\chi.$$

Then F-term is:

$$\int d^4x \left(\left[W(\Phi_i)\right]_F + c.c. \right) = \int d^4x \left(F\frac{\partial W}{\partial \phi} - \frac{1}{2}\chi\chi\frac{\partial^2 W}{\partial \phi^2} + c.c. \right)$$

The complete chiral superfield action is then:

$$S = \int d^4x \Big[\partial_\mu \phi^* \partial^\mu \phi + i\chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi + F^* F \qquad \leftarrow \text{D-term}$$

$$+ F \frac{\partial W}{\partial \phi} - \frac{1}{2} \chi\chi \frac{\partial^2 W}{\partial \phi^2} + \text{c.c.} \Big] \qquad \leftarrow \text{F-term.}$$

Eliminate auxiliary fields:

EOM for F and F^*:

$$\frac{\partial \mathcal{L}}{\partial F} = F^* + \frac{\partial W(\phi)}{\partial \phi} = 0 \qquad \Rightarrow \qquad F^* = -\frac{\partial W}{\partial \phi}$$

$$\frac{\partial \mathcal{L}}{\partial F^*} = F + \frac{\partial \overline{W(\phi)}}{\partial \phi^*} = 0 \qquad \qquad F = -\frac{\partial \overline{W}}{\partial \phi^*}$$

Plug back into action:

$$S = \int d^4x \Big[\partial_\mu \phi^* \partial^\mu \phi + i\chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi + \frac{\partial W}{\partial \phi} \frac{\partial \overline{W}}{\partial \phi^*}$$

$$- \frac{\partial \overline{W}}{\partial \phi^*} \frac{\partial W}{\partial \phi} - \frac{1}{2} \chi\chi \frac{\partial^2 W}{\partial \phi \partial \phi}$$

$$- \frac{\partial W}{\partial \phi} \frac{\partial \overline{W}}{\partial \phi^*} - \frac{1}{2} \chi^\dagger \chi^\dagger \frac{\partial^2 \overline{W}}{\partial \phi^* \partial \phi^*} \Big]$$

$$= \int d^4x \Big[\partial_\mu \phi^{*i} \partial^\mu \phi_i - \underbrace{\frac{\partial \overline{W}}{\partial \phi^{*i}} \frac{\partial W}{\partial \phi_i}}_{\text{scalar potential:}} + i\chi^{\dagger i} \bar{\sigma}^\mu \partial_\mu \chi_i - \frac{1}{2} \Big(\underbrace{\chi_i \chi_j \frac{\partial^2 W}{\partial \phi_i \partial \phi_j}}_{\text{Fermion-masses + scalar interactions}} + \text{c.c.} \Big) \Big]$$

$$V(\phi) = \frac{\partial \overline{W}}{\partial \phi^*} \frac{\partial W}{\partial \phi}$$

Indicies label superfield components:

If they furnish a representation of an internal symmetry group,

up index = conjugate rep. to down index

$$\Phi_i \rightarrow (e^{i\alpha^a T^a})_i{}^j \, \Phi_j$$